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Mathematical Model of Alcoholism Incorporating Treatment: A Case Study in Kenya

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ABSTRACT

Alcoholism is characterized by persistent and uncontrollable consumption of alcoholic beverages, which poses significant risks to physical, psychological, and emotional health, including conditions such as liver cirrhosis, epilepsy, cancer, hypertension, and diabetes. Furthermore, it contributes to substantial social and economic challenges, including road accidents, domestic abuse, unemployment, and elevated crime rates. In Kenya, 12.2% of the population engages in alcohol abuse, where 10.4% are afflicted by alcohol-use disorders, thereby constituting a pressing public health concern. This study introduces a deterministic mathematical model describing alcoholism, which integrates treatment and counselling components and is articulated through the Ordinary Differential Equations (ODEs) framework. The model evaluates the ramifications of treatment for alcoholism, with stability analysis executed through the Jacobian matrix methodology and sensitivity analysis employing normalized forward sensitivity techniques. The reproduction number R0 was ascertained utilizing the next-generation matrix approach, wherein $R_0 > 0$ indicates ongoing alcohol misuse within the susceptible population. Global stability analysis conducted through the Quadratic- Lyapunov method indicates that the Alcohol-Free Equilibrium (AFE) and Alcoholic Equilibrium (AE) are globally asymptotically unstable. Numerical simulations were done to forecast the impact of critical parameters, with these simulations underscoring the necessity of enhancing recruitment into the treatment compartment and minimizing relapse rates to render alcoholism manageable. Effective intervention strategies encompass public awareness initiatives, reduction of stigma, provision of incentives for treatment engagement, enhancement of treatment services, and the utilization of technological advancements for ongoing support. This research bears significant implications for the legislators and the Ministry of Health in formulating policies that establish a robust foundation for future endeavours aimed at controlling alcohol addiction.

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1. INTRODUCTION

Mathematical modelling constitutes a methodological approach employed to replicate real-world scenarios by utilizing mathematical equations, facilitating the anticipation of future behaviours by creating a streamlined representation of an authentic system [1]. This approach allows researchers to better understand complex systems by breaking them down into simplified, analyzable components [2], [3], [4], [5]. A model is universally regarded as a depiction of reality intended for analytical examination. In particular, mathematical models serve as tools to encapsulate systems under scrutiny in mathematical constructs, providing insights into the underlying mechanisms and relationships. These interrelations are typically articulated through various forms of equations (dynamic) or through governing principles organized as computational algorithms [6].

One application of mathematical modelling lies in addressing public health issues, including substance dependency. Alcoholism recognized as a treatable condition analogous to other forms of substance dependency, poses significant challenges worldwide. Estimates from the World Health Organization (WHO) indicate that around 237 million men and 46 million women globally grapple with issues related to alcoholism and its associated challenges [7]. In the context of Kenya, 12.2% (3,293,495 individuals) engage in alcohol abuse, while 10.4% (2,807,569 individuals) are diagnosed with alcohol use disorders [8]. Factors influencing alcoholism encompass social interactions, psychological stress, mental health status, age, ethnicity, and gender. By employing mathematical models, researchers can explore these factors quantitatively, helping to design effective interventions and predict the impact of various strategies on reducing alcohol-related problems [9], [10].

Addiction represents a chronic physiological alteration induced by alcohol misuse. This alteration complicates the cessation of drinking among alcoholics, with individuals attempting to abstain often experiencing withdrawal symptoms after the discontinuation of alcohol consumption [11]. Several models have been developed concerning alcoholism; they include A model by Bhunu [12] on alcoholism that subdivided the human population into susceptible (those who do not consume alcohol and have never consumed it), $S(t)$, those who consume alcohol but have not become alcohol dependent $D(t)$, alcohol consumers are those dependent on alcohol $A(t)$ and those recovered with or without treatment $R(t)$. The analysis of the reproduction number showed conditions under which supporting the encouragement of moderate drinkers to quit alcohol consumption leads to a decrease in alcoholism better than alcoholics only quitting. However, the model did not consider in detail the impact of treatment as a remedy for alcohol consumption. The population was given by : $N(t)=S(t)+D(t)+A(t)+R(t)$ [12]. The model did not include the treatment class before the recovery class. Mayengo's [13] Present study divides alcohol consuming population into three time-dependent population proportions: Light drinkers, Medium drinkers, and Alcoholics. This study considers the nondrinking population in Susceptible and Recovered. It also proposes the establishment of compulsory isolation treatment facilities as an intervention strategy, in which alcohol addicts will be recruited for compulsory rehabilitation. The numerical simulation suggests that the implementation of compulsory isolation treatment facilities is an effective intervention as it has direct effects on the targeted Alcoholic population, with immediate effects of decreasing the number of alcoholics. However, compulsory isolation creates stigmatization of alcoholics.

Muthuri et al. [14] Developed with six human population-based compartments and one media compartment. The population-based compartments are: S -Susceptible who have never used alcohol in their life, Sa− individuals exposed to media campaigns and have never used alcohol, L - Light drinkers who drink two to three drinks one or two times a week, H - Heavy drinkers who are dependent on alcohol, T -individuals under treatment or in the rehabilitation centres and Q -individuals who have stopped drinking permanently. The media compartment M - is the density of media campaigns. The model concluded that an effective media campaign model should be one in which those exposed to mass media do not drink alcohol. Numerical simulation shows that a higher treatment rate reduces those in the alcohol-addicted class. However, most data were estimated, and media advertisement is very expensive.

The relation between stress and alcohol treatment was studied in various studies that compare alcoholics receiving treatment to those who do not. Individuals categorized as entering treatment exhibited a heightened likelihood of recognizing their drinking challenges more promptly; they also presented with a greater incidence of alcohol dependence symptoms, encountered more stressors, and faced negative occurrences across diverse life domains, thereby increasing their probability of recovery from alcoholism [15]. Thus, it is imperative to integrate treatment considerations into the discourse on alcoholism to evaluate its influence on the dynamics of this addiction, which is crucial for the effective management of alcohol dependency. The treatment of alcoholics typically entails the admission of individuals into rehabilitation facilities where counselling sessions and various programs are conducted to facilitate the voluntary alleviation of alcohol addiction. Treatment is essential for the regulation of alcoholism. Consequently, this study seeks to evaluate the efficacy of treatment in mitigating alcohol dependency and to propose recommendations for the control of alcoholism based on therapeutic interventions.

2. MODEL FORMULATION AND DEVELOPMENT

The model entails the establishment of a mathematical framework wherein the effective population is represented as (N). The epidemiological groups consist of individuals categorized into potential drinkers S(t), moderate drinkers M(t), alcoholics A(t) undergoing treatment T(t), and those who have recovered R(t). These classes are represented by the equation $N(t) = S(t) + M(t) + A(t) + T(t) + R(t)$, where N(t) includes all categories of effective population. The susceptible population S(t) experiences an increase attributable to births at the rate π, relapses of moderate drinkers to potential drinkers at the rate of λ_2, and contributions from recovered individuals at the rate ρ. This compartment experiences a decrement at the rate λ_1 alongside natural mortality at the rate μ. The subsequent compartment pertains to the moderate drinkers class M(t), which is augmented at the rate λ_1 by susceptible individuals commencing alcohol consumption for the first time, at the rate r_2 by those relapsing from alcoholic status to moderate drinkers.

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In contrast, this compartment diminishes due to moderate drinkers reverting to potential drinkers at the rate λ_2. Those escalating their consumption from moderate to alcoholic drinkers at the rate r_1, culminating in fatalities at a rate of μ, individuals are lost due to natural causes. The third compartment encompasses individuals classified as Alcoholics A(t), defined as consumers who exhibit alcohol dependence.

The membership within this compartment is augmented by moderate drinkers who intensify their consumption patterns to alcoholic levels at the rate r_1, by individuals relapsing from the recovered status to alcoholic consumption at the rate β, and from those reverting from the treatment compartment to alcoholic status at the rate α_2. This compartment is diminished by individuals who succumb to health complications arising from alcohol consumption at the rate μ_1, those who die due to natural causes at the rate μ, and individuals transitioning to the treatment class at the rate α_1. The fourth classification pertains to Alcohol Consumers Undergoing Treatment and Counseling T(t). Individuals are incorporated into this class from the alcoholic category seeking treatment at the rate α_1. In contrast, this class experiences a decrement through transitions to the recovered class post-treatment at the rate ϕ, through deaths either resulting from treatment complications or occurring during the treatment phase at the rate μ_2, and those who perish due to natural causes at the rate μ, alongside relapses to alcoholic status from the treatment class at the rate α_2. The fifth compartment class encompasses those who have stopped drinking R(t), which includes recovered who have ceased consumption following rehabilitation or counselling in a recognized institution and are deemed to have stopped drinking R(t). Membership in this class is constituted by alcoholics who have undergone rehabilitation and been certified as having recovered from alcohol dependence at the rate ϕ. In contrast, the population in this class decreases due to natural mortality at the rate μ, fatalities occurring in the recovered class after cessation of alcohol consumption at the rate μ_3, relapses to alcohol consumption following treatment and certification of recovery at the rate β, and finally, transitions to potential drinkers from the recovered class at the rate ρ. Figure 1 represents the flow chart diagram.

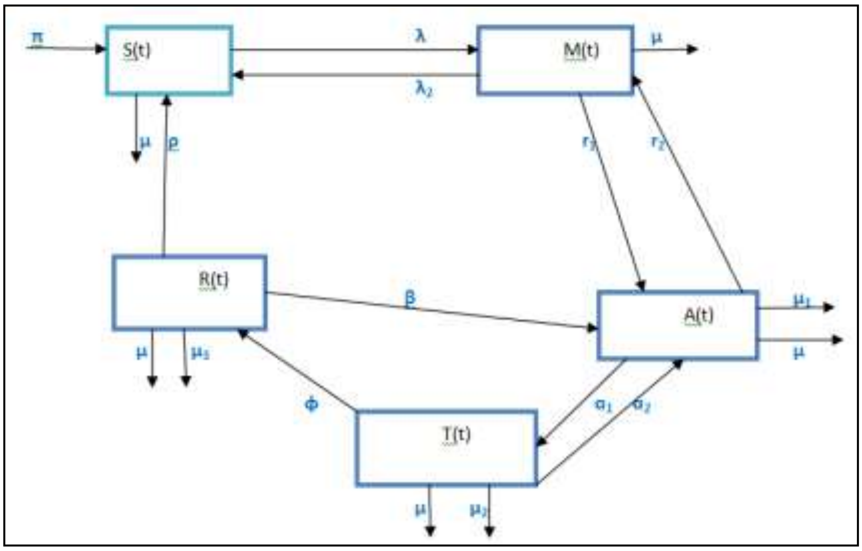

Figure 1. Flow chart diagram

Equation (1) represents the model equations.

$$\begin{cases} \frac{dS}{dt} = \pi + \lambda_2 M + \rho R - (\mu + \lambda)S \\ \frac{dM}{dt} = \lambda S + r_2 A - (\mu + \lambda_2 + r_1)M \\ \frac{dA}{dt} = r_1 M + \beta R + \alpha_2 T - (\alpha_1 + r_2 + \mu_1 + \mu)A \\ \frac{dT}{dt} = \alpha_1 A - (\alpha_2 + \phi + \mu_2 + \mu)T \\ \frac{dR}{dt} = \phi T - (\rho + \beta + \mu_3 + \mu) \end{cases} \quad (1)$$

The effective population is given by $N(t) = S(t) + M(t) + A(t) + T(t) + R(t)$ in reference to
$N(t) = S(t) + D(t) + A(t) + R(t)$ [12].
The force of infection, λ is given by the equation below:

$$\lambda = \frac{DC(\epsilon_1 M + \epsilon_2 A + \epsilon_3 T)}{N}$$

Where D represents non-drinkers' contact and C represents the probability rate of alcohol consumption.
The parameter values in the model are given in the Table 1.

Table 1. The parameter values in the model

symbol	parameter	values	Reference
π	Recruitment rate	0.0546	Vivas et al. [16]
λ_2	Rate of relapse from moderate drinkers to potential drinkers	0.7	Muthuri et al. [14]
r_1	Rate of recruitment from moderate drinkers to alcoholics	0.055	NACADA (2012)
r_2	Rate of relapse from Alcoholics to moderate drinkers	0.2	Manthey [17]
α_1	Rate of recruitment from Alcoholics to Treatment class	0.131	Vivas et al. [16]
α_2	Rate of relapse from Treatment class to Alcoholics	0.13	Miller et al. [18]
β	Rate of relapse from Recovered Drinkers to Alcoholics	0.001	Bhunu [12]
μ_1	Natural death rate that is not caused by alcohol in each class	0.02	Bhunu [12]
μ_2	Death rate related to drinking alcohol in the Alcoholic class	0.002	Sandow [19]
μ_3	Death rate related to treatment	0.035	Bhunu [12]
Ø	Recruitment to recovered class	0.5	Assumed
ρ	Recruitment from recovered class to potential drinkers	0.03	Tireito [20]
ϵ_1	Parameter related to moderate drinkers' effect on alcohol transmission	0.5	Assumed
ϵ_2	Parameter related to alcoholics' effect on alcohol transmission	0.5	Assumed
ϵ_3	Parameter related to treatment effect on alcohol transmission	0.7	Orwa [21]
D	A parameter related to the number of contacts between alcoholic and non-alcoholic	4.13	Mayengo [13]
C	A parameter related to the probability rate of consumption of alcohol	0.81	Cowling [22]
λ	$\lambda = \frac{DC(\epsilon_1 M + \epsilon_2 A + \epsilon_3 T)}{N}$	0.006	Assumed

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3. MODEL ANALYSIS

3.1. Positivity of the Solution

Since the model system (equation (1)) relates to living organisms, all variables must be non-negative for any given time $t > 0$. Consequently, with the given initial conditions, the solutions of the model (1) remain positive for all $t > 0$.

Theorem 3.1.
Given the initial conditions, $S(0) \geq 0, M(0) \geq 0, A(0) \geq 0, T(0) \geq 0$ and $R(0) \geq 0$, the solutions S(t), M(t), A(t), T(t), and R(t) of system (1) remain positive for all $t > 0$.

Proof. By solving the first equation of system (1) for S(t) at time $t > 0$, the following result is obtained:

$$\frac{dS}{dt} = \pi + \lambda_2 M + \rho R - \mu S - \lambda S$$

$$\int \frac{dS}{S} \geq -(\mu + \lambda)dt$$

$$\ln S(t) \geq -(\mu + \lambda)t + c$$

$$S(t) \geq e^{-(\mu+\lambda)t+c}$$

$$\text{Taking } (e^{c}) \text{ to be } (A) S(t) = Ae^{-(\mu+\lambda)t}$$

Thus,

$$S(t) \geq S(0)e^{-t(\mu+\lambda)}$$

Therefore, S(t) is non-negative, guaranteeing that S(t) remains positive over time. A similar approach can be used to establish the positivity of the other variables by utilizing their respective equations in the system. This shows that the solutions of system (2.1), starting with non-negative initial conditions, will remain non-negative for all $t \geq 0$, ensuring that $S(t) > 0$, $M(t) > 0$, $A(t) > 0$, $T(t) > 0$, and $R(t) > 0$

3.2. Invariant Region

A mathematical problem is considered well-posed if it has a unique solution, and its solution continuously depends on the initial data and parameters.

Theorem 3.2.
There exists a domain Y in which the solution set S(t), M(t), A(t), T(t), and R(t) of the model equation (1) is positively invariant.

Proof. The following equation can express the total human population:

$$N(t) = S(t) + M(t) + A(t) + T(t) + R(t) \tag{2}$$

Equation (2) defines the population as the sum of all the model variables. Next, the rate of change of N(t) along the trajectories of the model (1) yields equation (3):

$$\frac{dN}{dt} = \pi - \mu N - \mu_1 A - \mu_2 T - \mu_3 R \quad (3)$$

Without the presence of alcoholism in the population, the equation reduces to:

$$\frac{dN}{dt} = \pi - \mu N$$

and,

$$\frac{dN}{dt} + \mu N < \pi$$

Using the integrating factor $e^{\mu t}$ To solve:

$$N(t) < \frac{\pi}{\mu} + Ce^{-\mu t}$$

At t = 0

$$N(0) - \frac{\pi}{\mu} < C$$

Substituting:

$$N(t) \geq \frac{\pi}{\mu} + \left(N(0) - \frac{\pi}{\mu}\right) e^{-\mu t}$$

Where $N(0) = S(0) + M(0) + A(0) + T(0) + R(0)$ is the initial population. As $t \to \infty$, if:

$$N(0) \geq \frac{\pi}{\mu}$$

then,

$$N(t) \geq \frac{\pi}{\mu} \text{ as } t \to \infty$$

Therefore,

$Y = \left\{\left(S(t), M(t), A(t), T(t), R(t)\right) \in R_+^5 : N(t) \geq \frac{\pi}{\mu}\right\}$, This defines a valid solution region for the model equation (1), indicating that the total human population remains constant over time. As a result, the model is biologically meaningful and mathematically well-defined within the boundaries of region Y.

3.3. Basic Reproduction Number

The basic reproduction equation and its corresponding number quantify the potential for the transmission of alcoholism addiction. It evaluates the average number of new alcoholism cases that result from the introduction of a single alcoholic individual into a population susceptible to addiction. The basic reproduction number is calculated using the next-generation matrix (NGM), where the Jacobian matrix derived from the model equations is used to determine the reproduction rate.

Theorem 3.3.

The reproductive equation R_0, *for the epidemiological model of alcoholism, is expressed as;*

$$\mathcal{R}_0 = \frac{m\beta_0\left(k_2k_3\epsilon_1 + k_3r_1\epsilon_2 + \alpha_1(-\alpha_2\epsilon_1 + r_1\epsilon_3)\right)}{k_3r_1r_2 + k_1(-k_2k_3 + \alpha_1\alpha_2)}$$

According to Sepulveda et al. [23], as follows

were,

$$k_1 = \mu + \lambda_2 + r_1, k_2 = \alpha_1 + r_2 + \mu_1 + \mu, k_3 = \alpha_2 + \phi + \mu_2 + \mu, m = \frac{\pi}{\mu}, \beta_0 = DC$$

Proof. According to Sepulveda et al. [23], the Alcoholic subsystem is deduced from three equations:

$$\frac{dM}{dt} = \lambda S + r_2A - (\mu + \lambda_2 + r_1)M$$

$$\frac{dA}{dt} = r_1M + \beta R + \alpha_2T - (\alpha_1 + r_2 + \mu_1 + \mu)A$$

$$\frac{dT}{dt} = \alpha_1A - (\alpha_2 + \phi + \mu_2 + \mu)T$$

And the force of alcoholism taken as $\lambda = \frac{DC(\epsilon_1M+\epsilon_2A+\epsilon_3T)}{N}$ where

$$k_1 = \mu + \lambda_2 + r_1, k_2 = \alpha_1 + r_2 + \mu_1 + \mu, k_3 = \alpha_2 + \phi + \mu_2 + \mu, m = \frac{\pi}{\mu}, \beta_0 = DC$$

The Jacobian matrix J from the system becomes:

$$J = \begin{bmatrix} -k_1 + m\beta_0\varepsilon_1 & m\beta_0\varepsilon_2 + r_2 & m\beta_0\varepsilon_3 \\ r_1 & -k_2 & \alpha_2 \\ 0 & \alpha_1 & -k_3 \end{bmatrix}$$

The Jacobian matrix of the system is decomposed into two matrices, F and V.

$$F = \begin{bmatrix} -k_1 + m\beta_0\varepsilon_1 & m\beta_0\varepsilon_2 + r_2 & m\beta_0\varepsilon_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} 0 & 0 & 0 \\ r_1 & -k_2 & \alpha_2 \\ 0 & \alpha_1 & -k_3 \end{bmatrix}$$

Therefore, the basic reproduction number for the system model (1) is determined by calculating the spectral radius of the matrix FV^{-1}. Hence, the reproduction number is expressed as: $\mathcal{R}_0 > 1$*: Indicates* that alcoholism is spreading in the population. Parameters *D*, *C*, r_1, α_1, and α_2 can significantly influence whether the reproduction number R_0 is positive. A combination of increased contact rates (D), transmission probabilities (C), recruitment from moderate to alcoholic (r_1), and relapse rates (α_2) will push R_0 above 1, indicating that alcoholism will spread within the population. Each alcoholic influences more than one other person to either relapse into or start drinking. In this case, interventions (like treatment, education, and prevention) cannot control the problem. When $\mathcal{R}_0 < 1$: Suggests that the interventions (such as treatment programs, social policies, and awareness campaigns) are effective and that alcoholism is declining within the population. Each alcoholic influences fewer than one person to relapse or start drinking. In the context of Kenya's alcoholism crisis, this reproduction number helps quantify how quickly alcohol consumption behaviour spreads and whether the current measures (treatment, rehabilitation, awareness, etc.) are sufficient to control it. Understanding and minimizing R_0 can be key to reducing the prevalence of alcoholism in the country.

3.4. Stability Analysis

The local stability of the equilibrium points in the modelled system was analyzed to understand how the system responds to small disturbances. An equilibrium point is considered stable if, after a perturbation, all solutions converge to that point. Conversely, if the solutions do not converge, the equilibrium point is deemed unstable. To assess this, we will apply the Gershgorin Circle Theorem, which is presented in the following theorem:

Theorem 3.4.1. (Gershgorin Circle Theorem)
Let A be a $n \times n$ matrix with real entries. If the diagonal elements a_{ii} of A satisfy $a_{ii} < r_i$, where r_i represents the sum of the absolute values of the non-diagonal elements in the i^{th} row, Where,

$$r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

For i=1, 2,…,n, where r_i is the sum of the absolute values of the non-diagonal elements in the i^{th} row, then the eigenvalues of A are either negative or have negative real parts. The following corollaries are applied to analyze the stability of equilibrium points in the selected models:

Corollary 1**:** The disease-free equilibrium is locally asymptotically stable if $\mathcal{R}_0 < 1$. This means alcoholism increases with time reduces.

Corollary 2: The endemic equilibrium is locally asymptotically stable $\mathcal{R}_0 > 1$ implies that alcoholism increases with time, demanding intervention measures like treatment to reduce the increase of alcohol addiction.

3.4.1. **Alcohol-Free Equilibrium Point**

The Jacobian matrix J for the system of differential equations given by equation (1) where alcoholism is present, and force of infection in moderate, alcoholic, and treatment compartment is presented as follows:

$$\begin{pmatrix} k_1 & \lambda_2 - \epsilon_1 m\beta_0 & -\epsilon_2 m\beta_0 & -\epsilon_3 m\beta_0 & \rho \\ 0 & k_2 + \epsilon_1 m\beta_0 & r_2 + \epsilon_2 m\beta_0 & \epsilon_3 m\beta_0 & 0 \\ 0 & r_1 & k_3 & \alpha_2 & \beta \\ 0 & 0 & \alpha_1 & k_4 & 0 \\ 0 & 0 & 0 & \phi & k_5 \end{pmatrix}$$

Using,

$k_1 = -\mu; k_2 = -(\mu + \lambda_2 + r_1); k_3 = -(\alpha_1 + r_2 + \mu_1 + \mu);$

$k_4 = -(\alpha_2 + \phi + \mu_2 + \mu); k_5 = -(\rho + \beta + \mu_3 + \mu); \beta_0 = D \cdot C; m = \dfrac{\pi}{\mu}$

The Gershgorin Circle Theorem (GCT) asserts that every eigenvalue of a matrix lies within at least one of the Gershgorin discs, which are defined for each row of the matrix. For each row i of the matrix J, the corresponding Gershgorin disc D_i is centred at the diagonal element a_i, with a radius R_i equal to the sum of the absolute values of the non-diagonal elements in that row. $D_i = \{z \in C: |z - a_{ii}| \le R_i\}$

Where;

$$r_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

Calculating the Gershgorin Discs For the matrix J;

$$a_{11} = k_1 = -\mu, \quad R_1 = |\lambda_2| + |\epsilon_1 m\beta_0| + |\epsilon_2 m\beta_0| + |\epsilon_3 m\beta_0| + |\rho|$$

$$a_{22} = k_2 + \epsilon_1 m\beta_0, \quad R_2 = |r_2 + \epsilon_2 m\beta_0| + |\epsilon_3 m\beta_0|$$

$$a_{33} = k_3, \quad R_3 = |r_1| + |\alpha_2| + |\beta|$$

$$a_{44} = k_4, \quad R_4 = |\alpha_1|$$

$$a_{55} = k_5, \quad R_5 = |\phi|$$

Condition for Stability**:**

If all Gershgorin discs are entirely located in the left half of the complex plane (i.e., they do not touch or cross the imaginary axis into the right half), then all eigenvalues have negative real parts, indicating that the equilibrium is locally asymptotically stable.

Condition for Instability**:**

If any Gershgorin disc intersects or extends into the right half of the complex plane, the matrix has at least one eigenvalue with a positive real part, signifying instability.

If all k_i values (with $i \in \{1,3,4,5\}$) are sufficiently negative and the radii R_i are small, the Gershgorin discs will be entirely contained within the left half of the complex plane, indicating local stability.

The critical case arises with the second disc, where the term $k_2+\epsilon_1 m\beta_0$ could potentially shift the disc into the right half-plane if $\epsilon_1 m\beta_0$ becomes large enough. If this disc crosses the imaginary axis, the system may become unstable.

Since one of the Gershgorin discs might intersect or extend into the right half of the complex plane, the matrix will have at least one eigenvalue with a positive real part, signalling instability. As a result, the alcohol-free equilibrium (AFE) is locally asymptotically unstable.

3.4.2. **Global Stability Analysis of the AFE**

We use the quadratic Lyapunov function method to examine the global asymptotic stability of the system.

Theorem 3.4.2
Consider the autonomous dynamical system described by $x^* = f(x)$, where $x \in R^n$ is the state vector, and $f: R^n \rightarrow Rn$ is a continuously differentiable function. Let x^* be an equilibrium point of the system, meaning that $f(x^*) = 0$. Assume there exists a quadratic Lyapunov function $V(x): R_n \rightarrow R_n$ of the form:

$$V(\mathrm{x}) = (\mathrm{x} - \mathrm{x}^*)^T P(\mathrm{x} - \mathrm{x}^*)$$

Where P is a symmetric positive definite matrix, i.e., $P = P^T > 0$, and the following conditions are met:

1. **Positive Definiteness:** V(x) is positive definite, meaning $V(x) > 0$ for all $x \neq x^*$, and $V(x^*) = 0$.
2. **Negative Definiteness of the Derivative:** The time derivative of V(x) along the system's trajectories, given by $V(x) = dV(x)/dt = \nabla V(x).f(x)$, is negative definite, i.e., $\dot{V}(x) < 0$ for all $x \neq x^*$.

Conclusion:
The equilibrium point x^* is globally asymptotically stable if these conditions are satisfied. This implies that for any initial condition $x(0) \in R^n$, the solution $x(t)$ will converge to x^* as $t \rightarrow \infty$.

To perform the global stability analysis, we construct the Jacobian matrix for the model system (1) and evaluate it at the Alcohol-Free Equilibrium (AFE), as shown below, with the symbols retaining the same meaning as defined in Section 3.4.2:

$$\begin{pmatrix} k_1 & \lambda_2 - \epsilon_1 m\beta_0 & -\epsilon_2 m\beta_0 & -\epsilon_3 m\beta_0 & \rho \\ 0 & k_2 + \epsilon_1 m\beta_0 & r_2 + \epsilon_2 m\beta_0 & \epsilon_3 m\beta_0 & 0 \\ 0 & r_1 & k_3 & \alpha_2 & \beta \\ 0 & 0 & \alpha_1 & k_4 & 0 \\ 0 & 0 & 0 & \phi & k_5 \end{pmatrix} \quad (4)$$

We select a quadratic Lyapunov function in the form:

$V(x) = x^T Px$

where x is the state vector, $x = (x_1, x_2, x_3, x_4, x_5)$, x^T, and P is a symmetric positive definite matrix.

Assuming $P = I$, where I is the identity matrix, the Lyapunov function simplifies to:

$V(x) = x^2{}_1 + x^2{}_2 + x^2{}_3 + x^2{}_4 + x^2{}_5$

The time derivative of V(x) along the system's trajectories becomes:

$$\dot{V}(x) = \frac{d}{dt}(x^T Px) = \dot{x}^T Px + x^T P\dot{x}$$

Since $P = I$, this simplifies to:

$$\dot{V}(x) = \dot{x}^T x + x^T \dot{x} = 2x^T \dot{x}$$

Substituting $\dot{x} = Ax$, we get:

$$\dot{V}(x) = 2x^T Ax$$

Substituting the matrix A into the expression:

$$\dot{V}(x) = 2\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} \begin{pmatrix} k_1 & \lambda_2 & -\epsilon_1 m\beta_0 & -\epsilon_2 m\beta_0 & -\epsilon_3 m\beta_0 & \rho \\ 0 & k_2 + \epsilon_1 m\beta_0 & r_2 + \epsilon_2 m\beta_0 & \epsilon_3 m\beta_0 & 0 \\ 0 & r_1 & k_3 & \alpha_2 & \beta \\ 0 & \alpha_1 & 0 & k_4 & 0 \\ 0 & 0 & 0 & \phi & k_5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

This simplifies to

$$\begin{aligned} \dot{V}(x) = 2(&k_1 x_1^2 + \lambda_2 x_1 x_2 - \epsilon_1 m\beta_0 x_1 x_3 - \epsilon_2 m\beta_0 x_1 x_4 - \epsilon_3 m\beta_0 x_1 x_5 + \rho x_1 x_5 \\ &+ (k_2 + \epsilon_1 m\beta_0) x_2^2 + r_2 x_2 x_3 + \epsilon_3 m\beta_0 x_2 x_4 + k_3 x_3^2 + \alpha_2 x_3 x_4 + \beta x_3 x_5 \\ &+ k_4 x_4^2 + \phi x_4 x_5 + k_5 x_5^2) \end{aligned}$$

For the system to be globally stable $\dot{V}(x)$ The time derivative $\dot{V}(x)$ must be negative definite, meaning $\dot{V}(x) < 0$ for all $x \neq 0$.

Diagonal terms (stability along individual axes): If $(k_1), (k_2 + \epsilon_1 m\beta_0), (k_3), (k_4)$ and k_5 are all negative, the diagonal terms contribute negatively to $\dot{V}(x)$, which is required for stability.

Off-diagonal terms (cross-coupling effects): Terms like $(\lambda_2 x_1 x_2), (\rho x_1 x_5)$, and $(r_2 x_2 x_3)$ it can be positive or negative, depending on the values of λ_2, ρ, and r_2. However, if the diagonal dominance is strong enough (i.e., the magnitudes of $(k_1), (k_2 + \epsilon_1 m\beta_0), (k_3), (k_4)$ and k_5 are significantly larger than the magnitudes of the off-diagonal terms), the diagonal terms will dominate, making $\dot{V}(x)$ negative. The system is globally asymptotically stable provided that the following conditions hold: The diagonal elements $(k_1), (k_2 + \epsilon_1 m\beta_0), (k_3), (k_4)$, ($k_5$) are all negative (ensuring that the contributions from the diagonal terms are negative). The off-diagonal terms are either negligible in comparison to the diagonal terms, or they combine in such a way that the overall contribution to $\dot{V}(x)$ remains negative. If these conditions hold, $\dot{V}(x)$ decreases over time, ensuring that x(t) converges to the equilibrium point is reached as t approaches infinity, confirming global asymptotic stability. Otherwise, it is unstable.

4. NUMERICAL SIMULATIONS

A numerical simulation of the system model (1) was conducted on the impact of recruitment rate to Treatment, the Combination of Intervention Strategies, the impact of recruitment rate from moderate drinkers to alcoholics, and the impact of relapse rate to Alcoholics using parameter values derived from reported studies, current alcohol consumption trends in Kenya, and some estimated values to provide meaningful analysis for this study. The parameter values listed in Table 1 were used for the numerical simulations. The simulations were carried out throughout $0 \leq t \leq 100$ days, during which addiction is expected to have fully developed. A baseline population of 1,000 individuals was used to represent the effective population. The simulations were performed using PYTHON software, with JUPYTER as the integrated development environment (IDE), and the results are presented in graphical form.

4.1. Numerical simulation on the impact of recruitment rate on treatment.

Analyzing Figure 2, it is evident that increasing the rate at which alcoholics transition to treatment results in a higher and earlier peak in the treatment population, highlighting its importance in managing alcohol abuse and addiction. The higher the treatment rate, the lower the alcohol addiction. The lower the treatment rate, the higher the alcohol addiction. Several strategies can be implemented to enhance this transition rate. For instance, increasing accessibility to treatment programs through expanded facilities and reduced costs can encourage more individuals to seek help and reduce alcohol addiction.

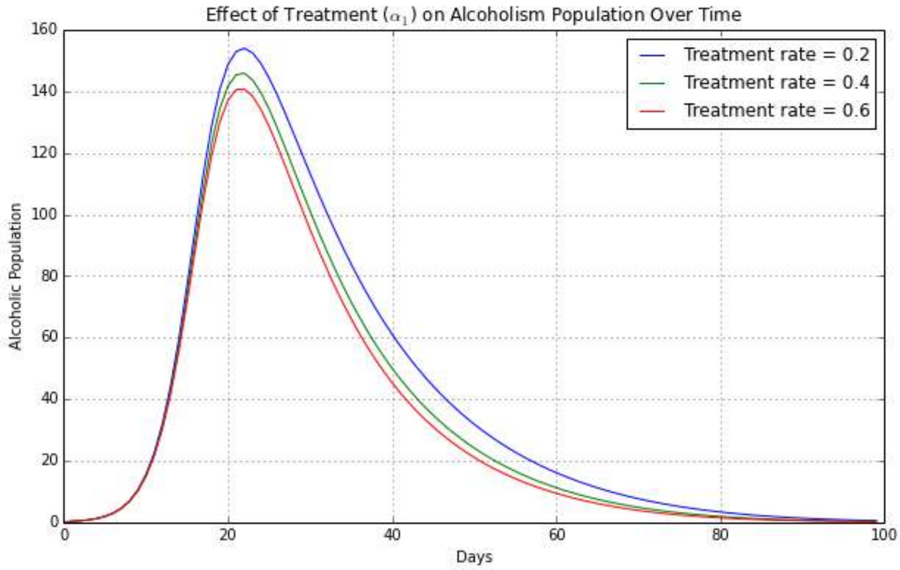

Figure 1. Effect of increase of transition rate of Alcoholics to Treatment.

4.2. Numerical Simulations on the Combination of Intervention Strategies.

Without treatment, alcoholism addiction generally leads to a progressive decline in physical and mental health, as well as increasing social and behavioural problems. Without the intervention of necessary strategies to lower the abuse of alcohol, the impact could be worse. Figure 3 illustrates this worsening trend over time, emphasizing the critical need for intervention measures of treatment to manage the addiction and prevent severe consequences.

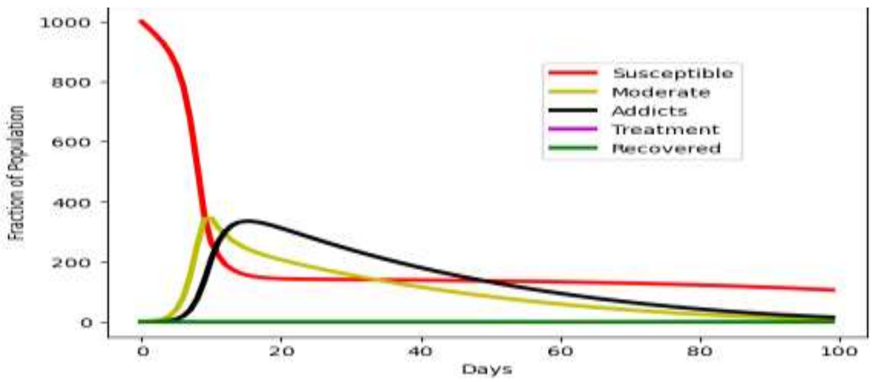

Figure 2. Impact of no Intervention Strategies.

The graph in Figure 4 presented below depicts the influence of intervention on the prevalence of addiction in a clear manner, showcasing how the implementation of a treatment strategy involving rehabilitation leads to a reduction from 350 alcohol abuse to almost 250 and increases recovered individuals to almost 150 by day 30 compared with a scenario without any form of intervention(treatment). Here, rehabilitation serves to decrease the population of individuals struggling with addiction. It does not affect those categorized as moderate drinkers. It is essential to emphasize the significance of implementing intervention measures aimed at reducing the pool of people with an addiction in order to reduce the probability of contact with moderate drinkers to reduce the reproduction number of alcoholics. Therefore, efforts should be concentrated on strategies that aim to limit the initiation of alcohol consumption among individuals by reducing the number of people with an addiction to a negligible level.

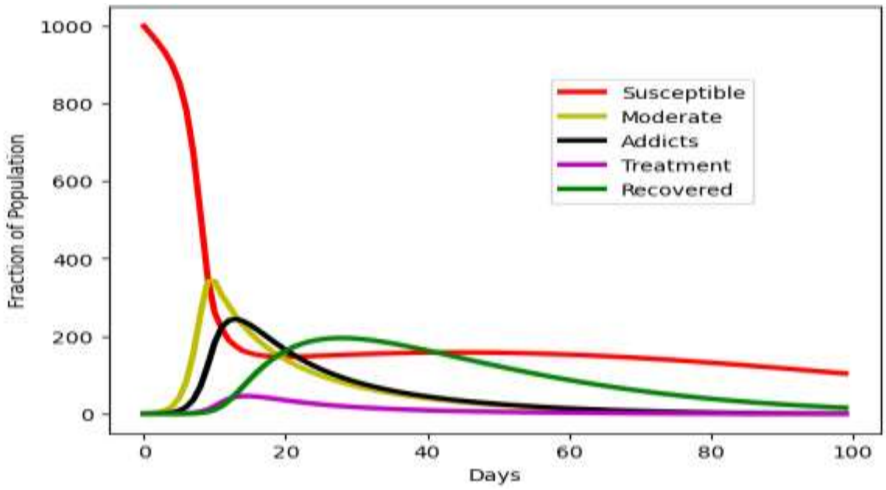

Figure 3. Impact of Intervention Strategies

4.3. Numerical Simulations on the impact of recruitment rate from moderate drinkers to alcoholics.

To minimize the impact of alcohol abuse in a community, efforts should focus on reducing the number of moderate drinkers, as indicated by Figure 5. Since the higher the recruitment of moderate drinkers to alcohol, the higher the rate of alcohol addiction, the lower the rate of recruitment, the lower the rate of addiction. Encouraging abstinence or minimal drinking can help lower the overall rate of alcoholic cases and prevent the escalation of alcohol-related issues and disorders.

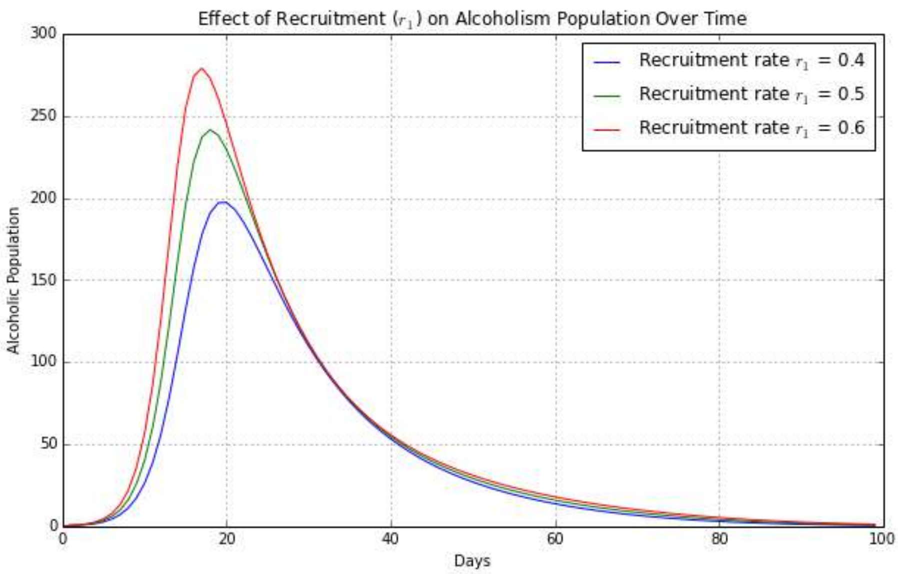

Figure 4. Impact of the rate of recruitment moderate to Alcoholics

4.4 Numerical simulation on the impact of relapse rate on Alcoholics

The relapse rates primarily drive the differences in the peak values of the treatment population. Lower relapse rates result in higher peaks because more individuals stay in the treatment compartment for longer periods, while higher relapse rates lead to lower peaks due to quicker transitions back to the alcoholic state, as shown in Figure 6.

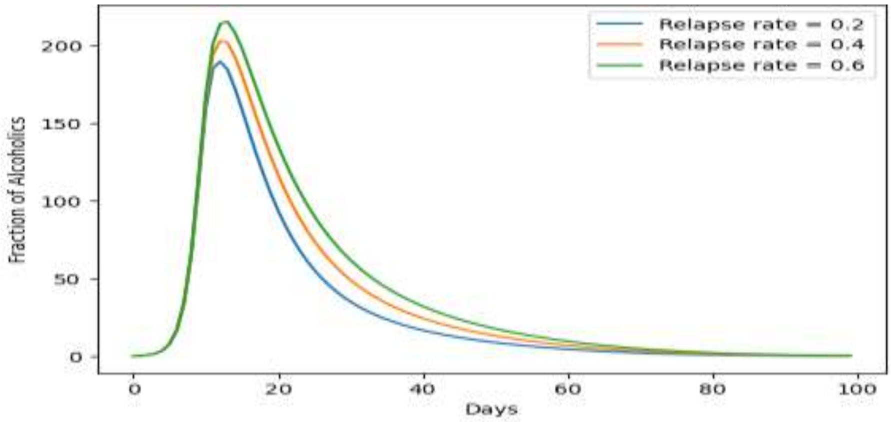

Figure 5. Impact of relapse rates.

5. DISCUSSION

The study developed the Susceptible-Moderate Alcoholic-Treatment-Recovered (SMATR) model, using Ordinary Differential Equations (ODEs) to analyze the transmission dynamics of alcohol addiction and assess interventions aimed at reducing addiction rates. This model's Alcohol-Free Equilibrium (AFE) and Alcoholic Equilibrium (AE) were evaluated for stability using the Jacobian matrix method, based on frameworks by Van Driessche and Watmough [24], with additional insights from Ochwach & Okongo [25]. The basic reproduction number (R_0), calculated via the next-generation matrix (NGM) method, was identified as a critical metric for understanding addiction persistence, where $R_0 > 1$ suggests addiction sustainability in the population [24]. Sensitivity analysis pinpointed key parameters influencing R_0, showing that policy enforcement (λ_2), treatment accessibility (α_1), and socioeconomic improvements (r_2) effectively curb addiction, while relapse rates (β) and social influences (r_1) increase it. These findings underscore the need for multifaceted interventions, including awareness campaigns, economic support, and strengthened support systems [25]. Additionally, the global stability analysis, achieved through the Quadratic Lyapunov function, confirmed the AFE's global asymptotic stability, indicating that targeted policies could potentially eliminate alcoholism on a large scale. Simulations revealed the importance of sustaining low R_0 and increasing treatment enrollment, highlighting that reducing stigma, providing incentives for rehabilitation, and implementing continuous support networks could be transformative. Improving treatment infrastructure, personalizing care plans, and integrating telemedicine further support long-term recovery, reducing relapse and promoting healthier communities. This model provides a comprehensive framework for policymakers and healthcare providers to address addiction sustainably.

6. CONCLUSION

This paper formulates a deterministic model to investigate the spread of alcohol addiction, considering the intervention measure of treatment. The model undergoes qualitative analysis and numerical simulations. The findings indicate that while treatment can lower the transmission rate of alcoholism, detailed plans are essential for rehabilitating alcoholics. Treatment also reduces peak days, providing early warnings for public health officials to expand rehabilitation facilities and minimize contact between alcoholics and non-alcoholics. Additionally, the study highlights the significant role relapse plays in the progression of alcoholism, emphasizing the need to reduce relapse rates through aftercare programs. It further notes that the rate of alcohol addiction is greatly influenced by the consistency with which alcoholics seek help and the accessibility of treatment facilities. Increasing treatment rates requires effective publicity and incentives, with government support to improve infrastructure and ease the burden on those seeking rehabilitation. Overall, the study concludes that treatment is an essential remedy for alcohol addiction, as it leads to faster outcomes and the reproduction number of alcoholism can be controlled or predicted for better planning.

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