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Inductive Insights into Preserving Euler Characteristics in Topological Transformations

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ABSTRACT

Topological transformations involve altering the shape or structure of surfaces without changing their fundamental properties. One key property is the Euler characteristic, a topological invariant that remains constant under continuous deformations. This study employs an inductive approach to explore how various transformations can preserve the Euler characteristic, providing insights into the underlying principles. The study aims to deepen our understanding of the relationship between topological transformations, changes in the number of vertices, edges, and faces, and their impact on Euler's characteristics. The study investigates the transformation of various objects, such as spheres and polyhedra, within the Euclidean plane, deriving generalizations about preserving Euler's characteristics. Furthermore, the article explores the ideas of proving Euler's theorems related to topological transformations in the Euclidean plane through an inductive approach. It focuses on the conceptual development of topological transformations, utilizing various visual representations and providing specific arguments. The article enhances the understanding of the concepts and principles involved in topological transformations while preserving Euler's characteristics by employing visual aids and illustrations. The article emphasizes explicitly the inductive derivation of the application of Euler's theorem in determining the genus of surfaces. The inclusion of visual representations and specific arguments further enhances the understanding and conceptual development of these topics.

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1. INTRODUCTION

Topological transformations of surfaces play a crucial role in understanding these geometrical objects' fundamental properties and characteristics. The term "topology" originates from the Greek words "topos," meaning "place," and "logos," meaning "study" or discourse [1]. Gauss first introduced it in the field of mathematics in 1847. It describes the study of properties preserved under continuous transformations, such as stretching or bending, but not tearing or gluing [2]. The etymological meaning of "topology" focuses on studying spatial relationships and the properties of spaces. Topologists analyze the fundamental characteristics of spaces, such as connectedness, continuity, and compactness, without relying on the notions of distance or measurement [2]. Topology concerns the properties of geometric objects preserved under deformations, such as stretching, squeezing, and bending but not tearing or gluing [3]. Topology studies the properties of surfaces under transformations like cutting, gluing, and welding [4]. Such topological transformations can alter a surface's fundamental characteristics while maintaining connectivity. A key property of surfaces is their Euler characteristic, which relates the number of vertices, edges, and faces in planar graphs that model the surface [5].

Euler made significant contributions to the development of topology. One of Euler's notable works that can be considered a precursor to topology is the Königsberg bridge problem. This problem laid the groundwork for studying networks and graph theory, which are closely related to topology. Euler's notable contribution to the field of topology is his formulation of the famous formula for polyhedra and Euler's characteristics. Euler's formula states that for any polyhedron, the number of vertices (V), edges (E), and faces (F) are related by the equation $V - E + F = 2$. This formula is a fundamental result in topology and has applications in various areas of mathematics and engineering.

Similarly, the development of Euler's characteristic (denoted by χ) for surfaces has provided valuable insights into the relationship between the topological structure of a surface and its geometric properties. Euler's characteristic, denoted by χ, is a topological invariant that describes the connectivity of a surface. Euler's formula states that for any polyhedron or two-dimensional manifold, the number of vertices (V) minus the number of edges (E) plus the number of faces (F) equals the Euler characteristic (χ): $V - E + F = \chi$ [6]. Euler's characteristic is a topological invariant, serving as a fundamental concept in topology. It has proven to be a powerful tool in understanding and classifying surfaces and establishing many significant mathematical results, ranging from the classification of regular polyhedra to the non-planarity criterion for graphs [7]. With the Euler characteristic of surfaces, mathematicians have gained valuable insights into the topological properties of surfaces, such as their genus (the number of handles or holes), face or region, edge or arc, and compactness. These insights have helped establish connections between the topological structure of a surface and its geometric properties, paving the way for further advancements in the field of topology [4].

This study aims to explore topological transformations of surfaces through an inductive approach, establishing rules for how operations like cutting, gluing, or welding change the Euler characteristic. Such insight could benefit mathematics education by concretely linking abstract topological concepts to spatial manipulations [8]. It may also support fields employing topological models, like chemistry, through a deeper mechanistic understanding of structural changes. This article explores the topic of topological transformations, their connection to Euler's properties, and the application of an inductive approach to analyze and comprehend the Euler's characteristic of a surface of genus P partitioned into F regions by V vertices joined by E arcs.

This approach allows for a more detailed and nuanced understanding of how Euler's characteristics are preserved in various scenarios through an inductive approach. It can provide more practical and applicable insights for real-world problems in geometry, physics, and other fields where topological transformations are relevant. This approach begins with specific instances of topological transformations and builds up to broader generalizations.

2. METHOD

The study is based on an inductive approach to explore the conceptualization of topological transformations of surfaces while preserving Euler's characteristics. This process involves examining specific examples, analyzing patterns and commonalities, identifying invariants to preserve the Euler characteristic, and iterating for a deeper understanding. It explores the transformation of various objects, such as spheres and polyhedra, within the Euclidean plane. This exploration aims to deepen the understanding of the relationship between topological transformations, changes in the number of vertices, edges, and faces, and their impact on Euler's characteristics. The study also focuses on the conceptual development of topological transformations and the ideas of proving Euler's theorems related to these transformations in the Euclidean plane. Utilizing an inductive approach, the researchers develop specific arguments and proofs for applying Euler's theorem in determining the genus of surfaces. To enhance the understanding and conceptual development of these topics, the article extensively employs various visual aids, including diagrams and illustrations obtained through deformation processes. This contributes to a deeper understanding of these fundamental concepts in topology and mathematics.

3. RESULTS AND DISCUSSION

3.1. Definitions of Fundamental Terms in Topological Transformations

Topological concepts are important for characterizing the types of shapes represented in geometric modeling systems. A basic introduction to topology can be found in references from the 1960s that describe homeomorphisms and topological invariance. While not rigorous, some intuitive topology definitions will be helpful here. A homeomorphism is a one-to-one and onto function between topological spaces with a continuous inverse, modeling elastic deformations that preserve neighbor relationships.

Connected Graph: A connected graph is a graph where all vertices are reachable from each other through paths in the graph. There are no isolated vertices or separated components.

Topological Surface: A topological surface is a 2D topological space that can be classified based on orientability, compactness, and genus as its distinguishing topological properties, such as Sphere, a torus (donut), a genus with g surface (g holes), Klein bottle, real projective plane, möbius strip, etc.

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Connected Surface: A connected surface is a topological surface where it is possible to continuously deform any path between two points to lie entirely within the surface without disconnecting it into multiple pieces. Connectivity is an essential requirement for classifying and studying surfaces.

Bounded Surface: A bounded surface is contained within an imaginary boundary and has a well-defined interior and exterior region, placing topological constraints on its global structure and properties, while an unbounded surface, like a plane, extends infinitely. A surface with a boundary separates itself from the rest by a closed curve. For example, a sphere, torus, disk, etc., are bounded surfaces.

Polyhedron: A polyhedron represents a solid 3D object built by combining planar polygon surfaces that connect at edges and vertices according to specific topological rules.

Simple Polyhedron: A simple polyhedron is a closed 3D shape with no topological holes in its surface. It has a single, unbroken boundary surface equivalent to a sphere.

Close Surface: A closed surface has no boundary and is a compact topological surface without edges or ends separating its interior from the exterior region like a sphere.

Open Disk: An open disk is the interior of a circle in 2D space, excluding the boundary. It includes the interior of a disk without its boundary circle.

Graph: A graph embedded in a surface cannot intersect except at vertices. It is an abstract data structure comprised of vertices and edges, which can model interconnected objects and their relations in discrete structures and networks.

Planner Graph: Planar graphs are embedded in planar surfaces like a plane. Unlike general non-planar graphs, it is a topological structure whose vertices and edges can be drawn in the 2D plane without any intersections.

Face: The face is the surface region. It is the region of the plane enclosed by edges in a planar embedding of a graph, with the external infinite region also constituting a face.

Region: A region refers to a subset of a bounded and connected surface. It denotes a connected portion of a surface outlined by curves and corresponds to faces bounded by edges and vertices. Every model contains a minimum of one region.

Genus: The genus of a surface is a measure of how many holes or handles it has. Genus quantifies the number of "holes" or handles in a closed topological surface, which is a key topological invariant. It relates to the minimum number of handles required to embed the surface on a sphere.

Torus: A torus is a topological space shaped like a donut or inner tube. It is compact and orientable, with Euler characteristic 0.

Vertex: A vertex in graph theory is a fundamental structural concept that refers to an edge's starting and ending point.

Edge: An edge connects two distinct vertices, representing boundaries between surface areas. These may be curved or straight-line segments.

Arc: An arc is a mathematical model for a line segment or continuous path between points in topology through its deformation properties.

Vertex Degree: A vertex's degree is the number of incident edges or adjacent edges for each vertex in a graph. It is the number of edges that are connected to that vertex.

Connected Face: A connected face has a single boundary, while a multiply connected face contains holes. Faces subdivide embedded graph surfaces into regions.

Manifold: A topological space that resembles Euclidean space near each point. Manifolds include surfaces like the Sphere, torus, plane, etc.

Two-manifold Surface: A two-manifold surface resembles a disk locally at each point. Manifolds may or may not be closed surfaces. This work focuses on two-dimensional manifolds.

Homomorphism: Homeomorphism is the condition of topological equivalence through continuous deformations that preserve incidence and boundaries. It is the characteristics of spaces that remain unchanged under continuous deformations. The meaning of homeomorphic in the Greek word 'homoios means identical, and 'morphe' means shape.

Orientability: Orientability refers to the possibility of choosing a globally coherent orientation (direction) everywhere on a topological surface in a continuous manner. It means a surface has two distinct sides, like a sphere and a non-orientable surface, like a Möbius strip, where the orientation cannot be continuously defined over the entire surface.

3.2. Defining Topological Transformations

Topological transformations refer to operations performed on geometric objects that preserve specific qualitative properties of the object [9]. They involve continuously deforming or manipulating an object through actions such as bending, stretching, squeezing, twisting, and similar actions without tearing or gluing [3]. These transformations play a crucial role in mathematics and various scientific disciplines by ensuring the maintenance of topological equivalence, allowing the transformed object to be continuously deformed back into its original shape. While discrete properties like distances and angles may change during the transformation, the fundamental connectivity of the object remains intact [3].

Examples of topological transformations include stretches, slides, spins, contractions, and expansions, as well as cuts, punctures, gluing, and welding for surfaces [8], [9]. Topology focuses on qualitative features such as connectivity, holes, and boundaries, providing a framework to study objects based on properties that remain invariant under bending or flexing motions [4], [5]. The application of topological transformations has been instrumental in classifying objects, establishing equivalence relations, and advancing understanding in diverse fields ranging from chemistry to data science [10].

Topological transformations are fundamental in the field of topology as they allow the study of shape without relying on properties like distance, enabling the establishment of shape correspondence between objects that can be continuously deformed into one another [9], [3]. These transformations have a significant impact on topological invariants, including

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the genus, Euler characteristic, and homology groups. The concept of topological equivalence through allowed transformations is at the core of topology. Topological transformations involve operations performed on surfaces that preserve their essential topological properties, such as the number of vertices, edges, and faces [3].

Topological transformations involve the continuous deformation of a surface without cutting or gluing, encompassing actions such as stretching, twisting, bending, and sliding portions of the surface [3], [2]. These transformations preserve important topological invariants of the surface [11]. Additionally, Euler's formula $V - E + F = 2$ represents a fundamental and stable formula for a class of geometric objects known as topological spheres. These objects possess specific topological properties. When a geometric object is made of an ideally plastic substance and undergoes deformation through stretching or contracting its parts while ensuring that different points do not coincide, it is considered to be under plastic deformation [11].

Moreover, the object can be cut and then reassembled after deformation [8]. Objects that undergo these allowed transformations are deemed topologically equivalent and share the same topological type [11]. Properties that remain consistent across all topologically equivalent objects are referred to as topological properties or topological invariants.

Transformations in topology are often formally expressed as homeomorphisms. Homeomorphisms establish a one-to-one correspondence between points in one topological space and points in another while preserving the topological structure or properties, such as connectedness, compactness, and the number of holes or handles. This idea captures the notion of topological equivalence [3]. Some examples of the homomorphic transformation of some objects are presented below graphically.

1. **Topological arc:** A topological arc is a continuous curve resembling a line segment with distinct endpoints. It is a fundamental concept in topology and is used to study the properties of curves and spaces (Figure 1).

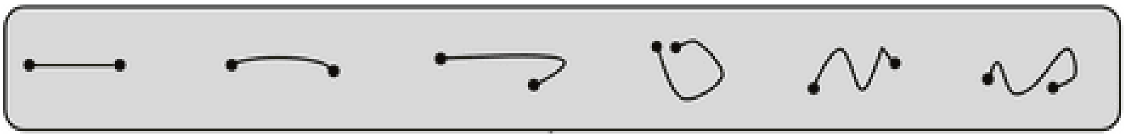

Figure 1. Topologically Equivalent Arc

2. **Topological circle:** A topological circle is a one-dimensional geometric object that is topologically equivalent to a circle. It retains the essential properties of a circle, such as being a closed curve with no endpoints, despite potentially being represented in a different geometric form (Figure 2).

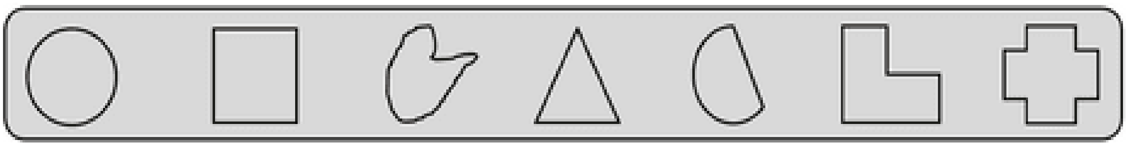

Figure 2. Topologically Equivalent Figures with Circle

3. **Topological disk:** A topological disk is a two-dimensional geometric object that is topologically equivalent to a disk or a closed surface with no boundary. It retains the essential properties of a disk, such as being connected and having a single boundary, even if its geometric representation may vary (Figure 3).

4. 

Figure 3. Topologically Equivalent Surfaces with the Disk

5. **Topological ring:** A topological ring is a two-dimensional geometric object that is topologically equivalent to a ring or a surface with a hole. It retains the essential properties of a ring, such as having a hole in the center, despite potential variations in its geometric representation (Figure 4).

Figure 4. Topologically Equivalent Surfaces with Ring

6. **Topological Sphere:** A topological sphere is a three-dimensional geometric object that is topologically equivalent to a sphere, having the properties of a closed, connected surface with no holes. It maintains the fundamental characteristics of a sphere, regardless of its specific geometric representation (Figure 5).

Figure 5. Topologically Equivalent Surfaces with the Sphere

3.2.1. Some Examples of Topological Transformations

Here are some examples of common topological transformations:

(i) Stretching/Squeezing a rubber sheet: This deforms the shape but maintains topological equivalence since it can be continuously morphed back without cuts or gluing.

(ii) Bending a paper strip into a circle: The strip is qualitatively unchanged as the circular form is continuously deformable back into a strip.

(iii) Twisting a coffee mug's handle: Rotating the handle 360 degrees leaves the mug topologically unchanged as original positions can be reached.

(iv) Cutting a torus along a meridian curve and regluing the boundary Produces another topologically equivalent torus through a Dehn twist transformation.

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(v) Contracting an edge in a graph: Removes a single edge but preserves the network's topology/connectivity.

(vi) Puncturing a sphere at a point: Results in another qualitatively identical sphere since the point can be continuously smoothed away.

(vii) Gluing opposite edges of a polygon to form a polyhedron: Transforms the planar tiling into a 3D ball or torus surface through identification.

(viii) Welding two Mobius strips together side by side: Yields a non-orientable surface with genus 2 through a discrete identification.

(ix) Edge flip in a triangulated surface: Locally modifies faces but preserves overall topological features.

These illustrate how connectivity and holes are preserved under continuous deformations or discrete operations. Various concrete examples are presented to illustrate topological transformations, emphasizing their impact on the Euler's properties of surfaces. These examples help elucidate how these transformations alter the geometric structure while preserving the topological characteristics.

3.4. Euler's formula for a simple closed polygon

Euler's formula relating the number of vertices, edges, and faces in planar graphs was one of his most important early contributions to graph theory. He published the foundational work in 1736 [12]. The concept of using vertices and edges to develop planar graphs on a flat surface without edge crossings originated from the Konigsberg Seven Bridges problem. This problem prompted Euler to define the concepts of planar graphs mathematically. He represented the number of vertices as V, the number of edges as E, and the number of faces as F, and developed the formula $V - E + F = 2$. This relationship, now known as Euler's formula for planar graphs or Euler's polyhedral formula, was the first topological invariant for planar graphs [12].

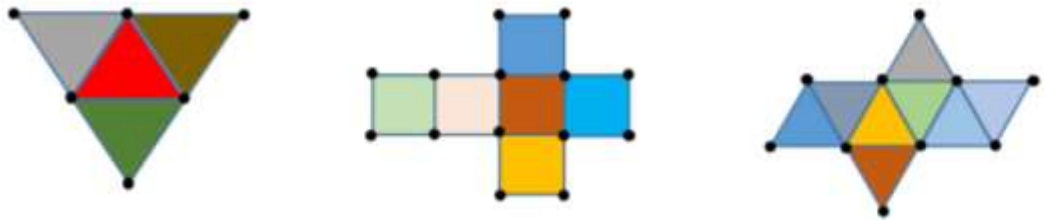
In later works, Euler generalized this formula to characterize surfaces of arbitrary topology [12], [13]. However, his 1736 paper established the foundation and was the original discovery of the relationship for planar graphs. This formulation introduced abstract graph theory and topological ideas into mathematics in a seminal way. Euler's formula remains a fundamental theorem regarding the structure of planar graphs [13].

Theorem 1: For any finite connected planar graph drawn without edge crossings, if V = vertices, E = edges, and F = faces/regions, then $V - E + F = 2$.

To prove this theorem, it is necessary to consider planar structures that can be drawn on a flat (planar) surface without edge crossings. In a planar graph or simple polygon, the Euler formula will hold as it provides a topological relationship between the basic components of a planar structure. Examples of planar structures include polyhedral faces, wiring diagrams, and transportation networks mapped on a 2D plane.

The number 2 on the right side of the Euler formula comes from the fact that any planar structure divides the plane into 2 semi-infinite faces. Any shape or structure drawn on a 2D plane divides the surface into regions called faces. There are always two special unbounded faces not fully enclosed by edges, known as semi-infinite faces - one inside and one outside the shape. These two faces extend forever in all directions, not bounded by edges, occupying the entire remaining area of the plane. Therefore, no matter the planar structure, two such semi-infinite faces will always be created by partitioning the plane.

This is why Euler's formula for a simple closed polygon uses the number 2 on the right side to account for these default faces representing the divided infinite plane. If we do not count the exterior unbounded face of the polygon or base part (2D drawing) as one region and only count the interior planar regions bounded by edges, we can use the modified Euler formula V – E + F = 1. For example, considering a planar graph drawn on a plane without edge crossings, the exterior unbounded region outside the polygon would not be counted as a face.



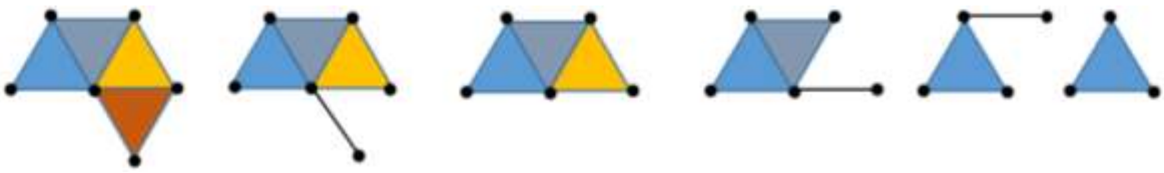
(i) (ii) (iii)

Figure 6. Planar Graph

Table 1. Table of Euler Characteristics of Planner Graphs

Figure	(i)	(ii)	(iii)
(χ)	$6-9+4=1$	$14-19+6=1$	$10-17+8=1$

In Figure 6, the value of the modified Euler formula V – E + F = 1 holds for all planar graphs of the tetrahedron, cube, and octahedron, where the exterior face is not counted (Table 1). In the 3D models of these polyhedra, where all faces (including the exterior infinite face) are counted, the value according to the standard Euler formula remains V – E + F = 2. Thus, the theorem can be proven inductively using a finite, connected, planar graph with a large number of vertices, edges, and faces. By removing one edge, both the edge count and face count are decreased by one in each edge removal step, while the value of V – E + F remains the same. By continuing this process until only a single face remains, the value of V – E + F remains unchanged. Hence, we can prove this theorem. Figure 7 illustrates the values at each step of edge removal, as presented in Table 2.



(i) (ii) (iii) (iv) (v) (vi)

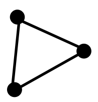
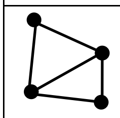
Figure 7. Step of edge removal

Table 2. Table of Euler Characteristics of Planner Graphs with Successive Edge Removal

Figure	(i)	(ii)	(iii)	(iv)	(v)	(vi)
(χ)	$6-9+4=1$	$6-8+3=1$	$5-7+3=1$	$5-6+2=1$	$4-4+1=1$	$3-3+1=1$

Similarly, it can be observed by initiating the graphs or polygons from one vertex or reversing the above process. Let us start by assuming one vertex in Table 3.

Table 3. Graphs of Polygons with Successive Vertex

Figures	**Descriptions**
	When we consider a graph or polygon with only one vertex, this single vertex contributes to the equation. Since there are no edges present, the edge count is zero. Therefore, the equation becomes: 1 – 0 + 0 = 1. (Without adding the outside region)
	Add another vertex to the given single vertex so that two vertex exist. Now, with two vertices present, the vertex count is two. Additionally, we have one edge connecting the two vertices. Therefore, the equation becomes: 2 – 1 + 0 = 1.
	Upon adding another vertex to the graph, we now have one distinct polygonal region (triangle). Now, we get one bounded region. With the addition of the new vertex, the total number of vertices becomes three. In this case, each vertex is connected to two other vertices, resulting in three edges. Therefore, the equation can be expressed as 3 – 3 + 1 = 1.
	By adding another vertex and joining it to the existing vertices with edges, the graph or polygon will create two faces, four vertices, and five edges. In this case, the equation becomes 4 – 5 + 2 = 1. Continuing this process, we can deduce the Euler formula for the simple polygon.

3.5. Euler's formula for the Sphere

The Euler formula for a simple polygon relates the number of vertices (V), edges (E), and faces (F) of a planar graph, given by V – E + F = 2. This formula was generalized in 1958 to spherical topology, known as Euler's formula for a simple polyhedron [13], [14]. Euler discovered that the relationship holds for convex polyhedra that are homeomorphic to the Sphere, establishing V – E + F = 2 as the formula describing spherical topology [13], [14]. The formula is also known as Euler's formula for the Sphere, with its special characteristic of 2. This formula can be applied to other polyhedra that are topologically equivalent to a sphere. However, the Euler characteristic of a regular polyhedron or a Platonic polyhedron is always 2, whereas other non-simple or non-regular polyhedra may have different characteristic values depending on their shape, number of sides, and regions. Before proving the theorem, some definitions are needed.

Definition 1: A polyhedron is a solid that is defined by its flat faces. The intersections of these faces form the edges of the polyhedron, and the vertices are the points where three or more edges come together.

Definition 2: A simple polyhedron is a type of polyhedron that is equivalent to a sphere in terms of its topology. It is a polyhedron that does not have any holes or voids in its structure.

***Definition 3*:** A regular polyhedron is a polyhedron whose faces are congruent regular polygons and where all its vertices have the same degree, meaning that the same number of edges meet at each vertex.

Definition 4: A non-simple polyhedron, also known as a complex polyhedron, is a polyhedron that has intersecting or self-intersecting faces or contains internal voids or holes in its structure. It does not meet the criteria of simplicity, where the faces are not allowed to intersect or overlap.

Definition 5: A non-regular polyhedron is a polyhedron that does not have congruent regular polygons as its faces or does not have vertices with the same degree.

Definition 6: Triangulation of a surface involves dividing it into a network of triangles using vertices and lines/arcs. This process creates a specific division of the surface into triangles. If a surface has non-triangular regions in its network, it can still be triangulated by adding diagonals/arcs from selected vertices. All triangulations of a given surface have the same Euler characteristic.

Theorem 2: For any simple polyhedron, $V – E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces.

First Method

This theorem can be proved in various ways. In this paper, the proof of Euler's characteristic formula for a sphere, $V – E + F = 2$, is based on the inductive method using triangulation. Triangulation involves creating triangles on the surface of the Sphere, where all the faces are triangles bounded by three edges. Triangulating a sphere is more straightforward than a sphere on Earth's surface, and it can be achieved by drawing the equator line and several lines of longitude from the North Pole to the South Pole, as shown in Figure 8.

Suppose we draw the equator line and n lines of longitude (let us consider $n = 4$). In this case, we obtain 2n triangular faces F ($2 \times 4 = 8$), n + 2 vertices V ($4 + 2 = 6$) [4 vertices on the equator line and 2 at the North and South Poles], and 3n edges E ($3 \times 4 = 12$), as illustrated in Figure 8 (ii). Therefore, Euler's characteristic formula $V – E + F = 2$ holds.

In other words, $(n + 2) – 3n + 2n = 2$, which simplifies to $6 – 12 + 8 = 2$.

Similarly, if we continue this process of triangulation on the surface of the Sphere, we obtain the same result, as shown in Figure 8 (iii) and (iv). In Figure 8 (iii), eight lines are joined to the equator line ($n = 8$), resulting in: $(n + 2) – 3n + 2n = 2 \Rightarrow (8 + 2) – 3 \times 8 + 2 \times 8 \Rightarrow 10 – 24 + 16 = 2$.

In Figure 8 (iv), twelve lines are joined on the equator line ($n = 12$), leading to: $(n + 2) – 3n + 2n = 2 \Rightarrow (12 + 2) – 3 \times 12 + 2 \times 12 \Rightarrow 14 – 36 + 24 = 2$.

This process of triangulation can be continued, and the formula will hold. However, it should be noted that each line should be drawn from the North Pole to the South Pole, passing through both hemispheres of the equator line. From Figure 8, we can also observe the relationship between vertices, edges, and faces on the surface of a sphere through the optimal triangulation of the Sphere's surface [15].

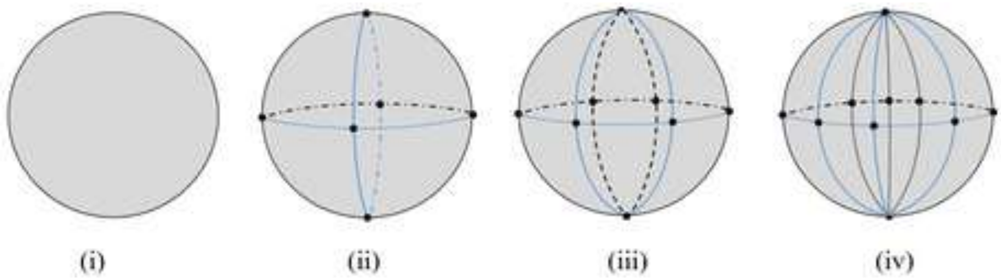


Figure 8. Triangulation of a Sphere

Second Method

This method is based on the technique of triangulation after removing one face of a simple polyhedron (cube) that is also topologically equivalent to a sphere. It utilizes the polyhedral surfaces in a connected 2D plane graph, employing theorem 1.

Let us consider a simple polyhedron, such as a hollow cube. We deform the cube by removing one face, resulting in a polyhedron with F faces, V vertices, and E edges. Next, we deform the remaining faces into a planar graph while placing the perimeter of the missing face externally. This deformation does not alter the number of vertices and edges but reduces the number of faces by 1. The objective is to prove V – E + F = 1 for this deformed, planar object, as depicted in Figure 9.

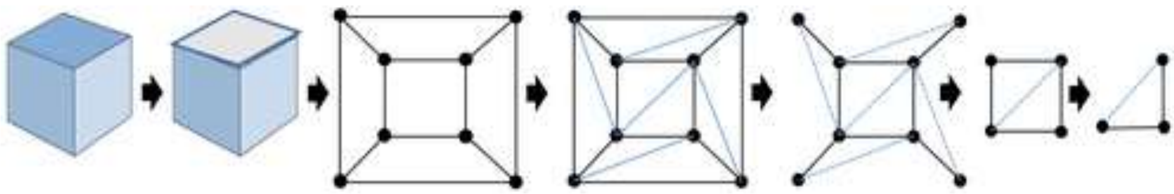


Figure 9. Triangulation of a Hollowed Cube

Next, we can proceed with triangulating each face of the deformed 2D planar object by drawing diagonals to connect two unconnected vertices in all planar regions. Adding each new diagonal introduces one edge and one face without changing the number of vertices. At this stage, the value of V – E + F is 2, including the removed face.

Continuing with the deformation process, we remove specific edges of boundary triangles that are not edges of other triangles. In each deformation step, the number of edges and faces decreases by one, but the number of vertices remains unchanged, preserving V – E + F.

As the deformation progresses, the planar graph eventually reduces to a single triangle. At this point, the triangle has V = 3, E = 3, and F = 1, satisfying V – E + F = 1. Since each deformation step preserves this quantity, it is demonstrated that V – E + F = 1 for the deformed, planar object. Therefore, it is also established that V – E + F = 2 (including the removed face) for the polyhedron (cube). The steps of deformation are illustrated in the figure. This completes the proof of the theorem.

Theorem 3: Two surfaces are topologically equivalent or homeomorphic if and only if their triangulations have the same Euler characteristic.

Theorem 4: There exist only five regular polyhedra or platonic solids.

3.6. Euler's Characteristic

Euler's formula establishes a connection between fundamental topological properties of polyhedra and surfaces using discrete quantities such as vertices (V), edges (E), and faces (F). The formula discusses key concepts and characteristics in topology that pertain to the topological properties of surfaces [15], [16]. The relationship between vertices (V), edges (E), and faces (F) is given by $V - E + F = \chi$, where χ represents the Euler characteristic, denoted by the Greek lowercase letter chi (χ). The Euler characteristic (χ) is an important topological invariant, which is determined by the number of vertices (V), edges (E), and faces (F) according to the well-known Euler characteristic formula: $V - E + F = \chi$ [5]. Topological transformations preserve the Euler characteristic since they do not alter the counts of vertices, edges, or faces [11].

The Euler characteristic χ is a topological invariant, meaning it remains unchanged under certain deformations of the shape or surface, such as stretching or twisting [3]. As stated in Theorem 2, the Euler characteristic of a sphere and a simple polyhedron is 2. This holds only for surfaces like the Sphere that have no holes or handles [3]. The Euler characteristic is dependent on the surface of the Sphere or polyhedron. For any polyhedron with a genus (g), the Euler characteristic (χ) is equal to $2 - 2g$, where g represents the genus. This equation is related to the number of "holes" or handles in the shape, so $\chi = 2 - 2g$ or $V - E + F = 2 - 2g$.

Definition 7: A surface is topologically equivalent to a sphere if and only if it has an Euler characteristic of 2.

3.7. Topological Transformations of Euler's Characteristic

Topological transformations play an important role in maintaining Euler's characteristic (χ) while allowing the discrete elements of a surface to change [3]. While topological transformations like stretching and twisting preserve χ, they can modify the number of vertices (V), edges (E), and faces (F) [5]. For example, during a deformation, discrete elements may be inserted or collapsed through operations such as vertex splitting, edge contraction, or face subdivision. Such discrete changes alter the values of V, E, and F, but the Euler's characteristics do not change [3]. Transformations visualized on digitized meshes demonstrate this property, as discrete operations alter triangle counts, but χ remains fixed [17].

Similarly, contractions in simplicial complexes maintain χ through cancellations in the $V - E + F$ summation [18]. Through topological equivalence, surfaces can be simplified while preserving their total genus and properties related to orientation, as captured by the Euler characteristic χ. This allows changes in discrete structure to be decoupled from the underlying continuous global characteristics, such as the Euler characteristic. Topological

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transformations can alter the values of vertices, edges, and faces while maintaining the overall Euler characteristic [3], [18].

3.8. Conceptualization of Topological Transformations through 3D Modeling

Topological transformation in 3D Modeling involves changing the topology of a geometric object while preserving certain properties [17]. A 2D disk can be transformed into various shapes using continuous deformation. It can become a cylinder, adding depth and curved surfaces. Rotating the disk around its axis creates a rounded shape called a torus. Topological transformations alter connectivity and structure without changing properties like genus or Euler characteristics [17], [19]. They allow for the creation of visually appealing 3D objects while preserving topological characteristics.

3.8.1. Disk 3D Modeling

Topological transformation in 3D Modeling involves changing the topology of a geometric object while preserving certain properties. A 2D disk can be transformed into various shapes using continuous deformation. It can become a cylinder, adding depth and curved surfaces. Rotating the disk around its axis creates a rounded shape called a torus. It is important to note that topological transformations can alter the connectivity, shape, and overall structure of a geometric object [20]. However, they do not change certain intrinsic properties, such as the genus (the number of holes) or the Euler characteristic (the relationship between vertices, edges, and faces) of the object. These topological properties remain invariant throughout the transformation process. They allow for the creation of visually appealing 3D objects while preserving topological characteristics (Figure 10).

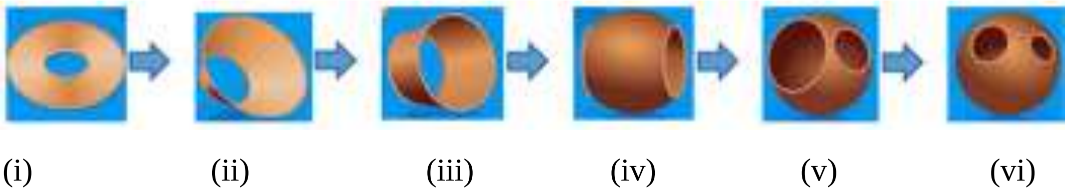

Figure 10. Disk 3D Modeling

3.8.2. Goldberg Sphere 3D Modeling

In Figure 11, the Sphere can be deformed into the shape of a Goldberg polyhedron using 3D modeling tools. Through an iterative process, the original Sphere can be transformed into a regular polyhedron known as a dodecahedron, which has a pure pentagonal mesh. Consequently, the dodecahedron exhibits the characteristic property of regular polyhedra, namely, the Euler's formula, which states that $V - E + F = 2$, where V represents the number of vertices, E represents the number of edges, and F represents the number of faces.

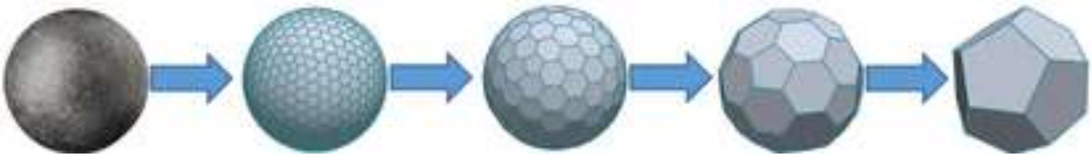

Figure 11. Goldberg Sphere 3D Modeling

3.8.3. Ico Sphere 3D Modeling

In Figure 12, the Sphere can be deformed into the shape of an Icosphere, and by iterating this process, the original Sphere can be transformed into a regular polyhedron known as an icosahedron, which has a pure triangular mesh. Consequently, the icosahedron exhibits the characteristic property of regular polyhedra, namely, the Euler's formula, which states that $V - E + F = 2$, where V represents the number of vertices, E represents the number of edges, and F represents the number of faces.

Figure 12. Ico Sphere 3D Modeling

3.8.4. Quad Sphere 3D Modeling

In Figure 13, the Sphere can be deformed into the shape of a Quad Sphere using 3D modeling tools. Through an iterative process, the original Sphere can be transformed into a regular polyhedron known as a cube, which has a pure rectangular mesh. Consequently, the cube exhibits the characteristic property of regular polyhedra, namely, the Euler's formula, which states that $V - E + F = 2$, where V represents the number of vertices, E represents the number of edges, and F represents the number of faces.

Figure 13. Quad Sphere 3D Modeling

3.9. Applications of Euler Characteristics of a Surface

The Euler characteristic is a topological invariant that provides valuable information about the structure and properties of surfaces. It helps examine specific examples of transformations such as stretching, twisting, and contraction/expansion of edges/faces on basic surfaces like the Sphere, genus, etc. [3]. By observing these transformations, one can see how the discrete counts of vertices (V), edges (E), and faces (F) change while the Euler characteristic χ remains the same [20]. This pattern reveals that transformations preserve χ through cancellations in the $V - E + F$ formula, keeping the difference constant [3]. This principle can be used to categorize new spaces, classify deformations, and gain a predictive understanding of topology through transformations [8]. Understanding how different objects transform and how they affect the number of vertices, edges, and faces in topological transformations provides insights into Euler's characteristics, as shown in the above figures. Before proving the theorem related to genus, it is helpful to discuss some examples of the topological properties of a sphere and handles in order to provide a basis for generalization.

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3.9.1. Deformation of Sphere by Making Two Hole

Let us consider a surface with two holes. The Sphere can be topologically transformed into an open cylinder, as shown in Figure 14. Now, we can divide the cylinder into a network of regions. It is important to note that the way we divide the cylinder is arbitrary. In Figure 14, the cylinder has been divided by joining the three lines AB, CD, and EF.

We know that the Euler characteristic of a cylinder is 2. In Figure 14, the cylinder is divided into n (3) regions by the lines AB, CD, and EF, with 2n (6) vertices and 3n (9) edges. Thus, its Euler characteristic is given by $2n - 3n + n = 0$. This is because two circular regions are missing due to the hollow cylinder. Therefore, it is shown that the Euler characteristic of a sphere with two holes is 0, which can be expressed as $V - E + F = 2 - 2 = 0$. This result indicates that each hole in a sphere corresponds to one face in its network representation. If a sphere has 2p holes, then what will happen?

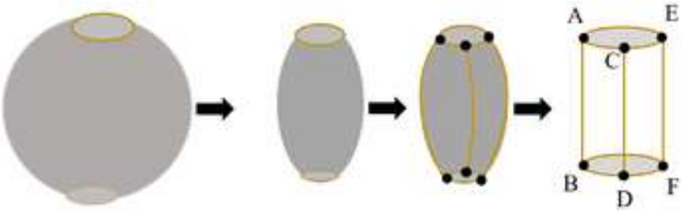

Figure 14. Deformation of Sphere by Making Two Hole

3.9.2. Removal of Handles of a Sphere

Let us consider a sphere with one handle (Figure 15), and we want to remove it from the surface while keeping the Euler characteristic of the surface unaltered. To achieve this, we will cut the handle at points A_1 and A_2 on the surface of the Sphere and remove the handle. After removal, the handle will have two free edges bounded by the new curves A_1* and A_2*. It is important to note that the number of vertices and arcs in the handle, represented by points A_1 and A_2, respectively, remains the same.

By removing the handle, the Sphere remains the same number of vertices, edges, and faces as before. Therefore, the Euler characteristic, $V - E + F$, of the surface remains unchanged. The additional vertices A_1* and A_2* introduced by cutting the handle balance out the additional arcs created, ensuring that the equation $V - E + F$ remains the same.

Furthermore, by removing the handle, no new regions or faces are created on the surface. The existing regions of the Sphere are unaffected, maintaining the same configuration and number of regions.

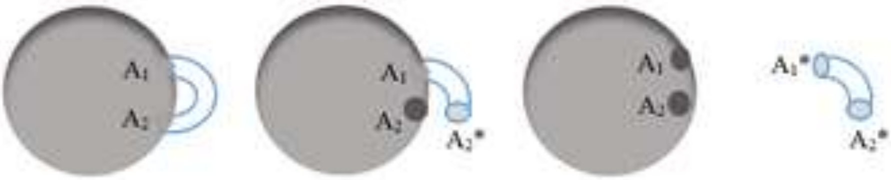

Figure 15. Removal of Handles of a Sphere

By doing this, we obtain two holes in the surface of the Sphere after cutting out the handle. A sphere with one handle produces two non-adjacent holes when the handle is removed from the surface. Similarly, a sphere with two handles results in four non-adjacent holes. In general, a sphere with n handles creates 2n holes on the surface after removing the handles while still maintaining the Euler characteristic of the original surface of the Sphere, thus preserving its topological properties. Similarly, a sphere with p handles creates 2p non-adjacent holes on its surface. However, 2p circular faces are required to maintain the holes of the Sphere with Euler characteristic V − E + F = 2 or the Sphere with 2p holes decreases 2p circular faces.

Theorem 5: If a surface of genus P is partitioned into F regions by V vertices joined by E arcs, then V − E + F = 2 − 2p

The theorem has been proved to connect with the relation of handles and holes of the Sphere presented above.

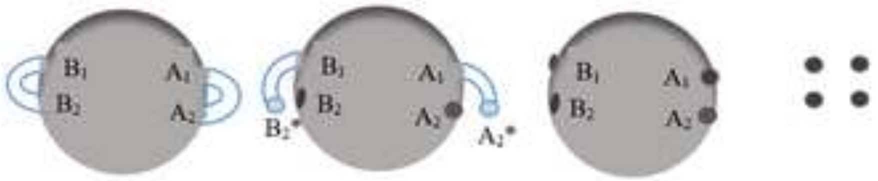

Figure 16. Removal of 2P Handles of a Sphere

Let us consider a sphere with P (2) handles (let p = 2), as shown in Figure 16. We can assume that the handles join the Sphere along closed curves, denoted as A_1, A_2, B_1, B_2, consisting of arcs from the given network, as shown in Figure 16. Now, let us cut off the handles along the curves A_1, A_2, B_1, B_2, on the surface of the Sphere. As a result, we obtain a sphere with 2p (2×2) holes, with the curves A_1, A_2, B_1, B_2, defining the boundaries of these holes and vertices that remain the same number of vertices, edges, and faces as before. Significantly, this deformation does not alter the Euler characteristic of the Sphere.

As discussed earlier, a sphere with 2 non-adjacent holes can be transformed into a hollow cylinder with an Euler characteristic of 0. We can generalize this concept to a sphere with p handles, which can form 2p non-adjacent holes on its surface. These 2p non-adjacent holes can form p hollow cylinders on the surface of the Sphere. The number of edges forming the boundaries of the holes and the number of regions into which the handles are divided can be arbitrary.

In this deformation process, two circular regions (faces) of the Sphere are lost for each handle. Consequently, the value of the Euler characteristic (V – E + F) is reduced by 2. If we remove four non-adjacent faces or two handles, the Euler characteristic decreases by 4. By generalizing this process, when p handles are removed, 2p non-adjacent holes are created on the surface of the Sphere. Therefore, we can conclude that the Euler characteristic of a sphere with p handles, forming 2p holes, is given by V – E + F = 2 – 2p. This is because a sphere without any missing faces has an Euler characteristic of 2 (V – E + F = 2). When we remove 2p non-adjacent faces while preserving their boundaries, the value of F in the Euler characteristic decreases by 2p. This completes the proof of the theorem.

4. CONCLUSION

Topological transformations of surfaces provide a powerful framework for understanding geometric properties while preserving essential topological characteristics. Incorporating Euler's properties enhances our comprehension of the relationship between topological transformations and changes in the number of vertices, edges, and faces. This study has utilized an inductive approach to deeply conceptualize the topological transformation of objects such as spheres and polyhedra, as well as their related theorems in the Euclidean plane, to derive generalizations about the impact of these transformations on Euler's properties.

The insights gained from this study have practical applications across diverse fields, including computer-aided design, architecture, and computational geometry. Understanding topological transformations and Euler's properties contributes to advancements in these domains, enabling more efficient and robust designs, simulations, and analyses. By applying the principles of topology and Euler's characteristics, researchers and practitioners can explore and manipulate geometric objects in a reliable and mathematically grounded manner. The study of topological transformations and Euler's properties not only deepens our understanding of the underlying principles of geometry but also empowers us to solve complex problems and innovate in various fields, leading to advancements and progress in both theoretical and practical realms.

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COMPETING INTERESTS

We want to make it clear that we have no competing interests.

ETHICAL STATEMENT

The study does not involve any human participants. It is based on a review of related literature and adheres to the code of conduct for citations.

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