Journal of Mathematics Instruction, Social Research and Opinion
Vol. 5, No. 3, September 2026, pp. 2415 – 2430, https://doi.org/10.58421/misro.v5i3.1537
ISSN 2962-7842

From Students' Ways of Thinking to Meaning Construction: Hermeneutics Perspective of the $\varepsilon - \delta$ Definition

Aditya Prihandhika[1], Mulia Putra[2]
[1,2]Universitas Singaperbangsa Karawang, Karawang, Indonesia

Article Info

Article history:

Received 2026-05-02
Revised 2026-05-31
Accepted 2026-08-13

Keywords:

$\varepsilon - \delta$ Definition
Differential Calculus
Hermeneutics Interpretation
Introductory Real Analysis
Mathematical Meaning
Students' Ways of Thinking

ABSTRACT

The $\varepsilon - \delta$ definition is widely recognized as one of the most conceptually demanding topics in differential calculus and introductory real analysis, as students often rely on intuitive notions of closeness without fully grasping its formal logical structure. Building on prior findings on students' ways of thinking, this study investigates how such ways of thinking contribute to the construction of mathematical meaning related to the $\varepsilon - \delta$ definition. This study employed a qualitative hermeneutic design involving written responses, task-based interviews, and reflective memos from 28 undergraduate students enrolled in calculus and introductory real analysis courses. Data were analyzed through a hermeneutic cycle consisting of naïve reading, selective highlighting, thematic interpretation, and the integration of students' interpretations with the formal mathematical meaning of the $\varepsilon - \delta$ definition. The findings reveal three dominant interpretive themes: distance meaning, where ε and δ are understood as measures of closeness; responsive dependency, where δ is interpreted as a response to a chosen ε; and control meaning, where δ functions as a mechanism for regulating input proximity. Persistent challenges include equality, dependency and confusion in quantifier sequencing. This study contributes by extending students' ways of thinking into a hermeneutic account of meaning construction, providing an epistemic basis for future didactic transposition studies. The findings also suggest instructional approaches that explicitly address distance, dependency, and logical sequencing in teaching the $\varepsilon - \delta$ definition.

This is an open-access article under the CC BY-SA license.

Corresponding Author:

Aditya Prihandhika
Faculty of Teacher Training and Education, Universitas Singaperbangsa Karawang
Email: aditya.prihandhika@fkip.unsika.ac.id

1. INTRODUCTION

The $\varepsilon - \delta$ definition remains one of the most conceptually demanding topics in differential calculus and introductory real analysis because students frequently rely on intuitive notions of closeness while struggling to coordinate the formal logical relationship between ε and δ [1], [2]. At an intuitive level, many students interpret limits as processes of getting closer, which leads to a predominantly dynamic and informal understanding that is difficult to reconcile with the static and quantified structure of the formal definition [3], [4]. This tension becomes particularly evident when students are required to interpret the quantified statement $\forall\varepsilon > 0 \ \exists \ \delta > 0$, where the logical sequencing and relational structure between ε and δ must be precisely understood [5], [6].

Empirical observations consistently show that students often treat ε and δ as independent or numerically equal quantities, rather than recognizing δ as a responsive value determined by ε [7]. As a result, the dependency structure central to the definition is frequently reduced to procedural manipulation without conceptual grounding [8]. Furthermore, students tend to interpret inequalities such as $|f(x) - L| < \varepsilon, |x - a| < \delta$ in a static manner, focusing on algebraic transformations rather than their underlying meaning as constraints on distance within neighborhoods, a phenomenon aligned with the dominance of procedural over conceptual understanding in mathematical learning [9], [10]. From a historical perspective, the formalization of limits through $\varepsilon - \delta$ definitions itself emerged as a response to the need for rigor beyond intuitive notions of infinitesimal change, highlighting the epistemological gap that students continue to experience today [11], [12]. Consequently, students' ability to construct a coherent interpretation of the $\varepsilon - \delta$ definition is hindered not only by its symbolic complexity but also by the absence of a meaningful bridge between their ways of thinking and the formal logical structure [13].

Our previous study on students' ways of thinking identified several recurring patterns, including proximity thinking, static symbolic reading, and equality dependency between ε and δ [14], [15]. Proximity thinking was characterized by students' tendency to interpret limits primarily as processes of getting closer, reflecting a dynamic concept image that has been widely documented in the literature on limit understanding [16]. In addition, static symbolic reading emerged when students treated the formal expression $\forall\varepsilon > 0 \ \exists \ \delta > 0$ as a fixed algebraic statement to be manipulated, rather than as a meaningful logical structure requiring interpretation, a phenomenon consistent with empirical findings that students often privilege procedural manipulation over conceptual understanding [17]. Furthermore, the pattern of equality dependency revealed that many students assumed a direct numerical correspondence between ε and δ, frequently expressing δ as equal to or proportionally identical with ε, which aligns with reported difficulties in recognizing the functional dependency of δ on ε [18].

Previous studies have successfully identified various ways of thinking that students employ when reasoning about the $\varepsilon - \delta$ definition, including proximity thinking, symbolic reading, and equality dependency [19]. While these studies provide valuable descriptions of students' reasoning patterns, they have primarily focused on identifying what students think and the difficulties they encounter. Considerably less attention has been given to how such ways of thinking become meaningful mathematical understandings [20]. In particular, limited research has examined how students interpret key elements of the $\varepsilon - \delta$ definition, such as distance, the relationship between ε and δ, and logical sequencing as meaningful mathematical constructs rather than merely formal symbols [21].

Consequently, there remains a need to move beyond the identification of reasoning patterns toward an interpretive account of how mathematical meaning emerges from students' existing ways of thinking. This limitation reflects a broader challenge in mathematics education research. Although studies on limits and formal definitions have extensively documented misconceptions, cognitive obstacles, and procedural tendencies, they rarely address how students negotiate meaning between their intuitive understandings and formal mathematical structures [22]. As a result, the gap between concept image and concept definition remains insufficiently understood as an interpretive process through which meanings are constructed, refined, and reorganized [23]. Addressing this issue requires a perspective that goes beyond descriptive analyses and examines how students' ways of thinking function as interpretive horizons through which formal concepts become meaningful [24].

To address this gap, this study adopts a hermeneutic perspective that views students' responses not merely as indicators of correct or incorrect reasoning, but as texts of meaning through which mathematical understanding is constructed [25]. Students' written and verbal expressions are treated as meaningful articulations shaped by prior experiences, intuitions, and conceptual resources. This perspective is grounded in philosophical hermeneutics, which emphasizes that understanding emerges through an ongoing dialogue between the interpreter and the text within historically situated horizons of meaning [26]. Central to this approach is the hermeneutic circle, in which interpretation moves recursively between parts and whole. Individual student statements (e.g., interpretations of ε, δ, or inequalities) are understood in relation to the broader conceptual structure of the $\varepsilon - \delta$ definition, while the meaning of the whole is simultaneously refined through attention to these local expressions. This recursive movement enables researchers to connect particular student responses with broader mathematical meanings, creating an interpretive meeting between students' understandings and the formal meaning of the $\varepsilon - \delta$ definition. Such an iterative process leads to what Gadamer terms a fusion of horizons, in which students' intuitive and experiential understandings are brought into dialogue with the formal mathematical horizon of the definition. Within this fusion, meanings such as distance, dependency, and control are reconstructed through interpretation rather than imposed externally [27].

By adopting this lens, the study shifts the focus from viewing students' ways of thinking as fixed cognitive states toward understanding them as dynamic interpretive resources that mediate the construction of formal mathematical meaning. Such a perspective allows for a more nuanced account of how students negotiate the tension between informal intuition and formal symbolism, thereby providing deeper insight into the processes through which the $\varepsilon - \delta$ definition becomes meaningful. This hermeneutic orientation thus offers a powerful framework for bridging the gap between students' existing ways of thinking and the formal logical structure of advanced mathematical concepts.

By clarifying how students' ways of thinking evolve into structured mathematical meanings, this study is expected to contribute both theoretically and practically. Theoretically, it extends ways of thinking research by providing a hermeneutic account of meaning construction. Practically, it offers insights for the teaching of limits and introductory real analysis by highlighting how notions of distance, responsiveness between ε and δ, and logical sequencing can be explicitly addressed within instructional design. Accordingly, this study aims to interpret how students' existing ways of thinking contribute to the construction of meaning related to the $\varepsilon - \delta$ definition. This study addresses three questions: (1) How do students construct the meaning of the $\varepsilon - \delta$ definition from their existing ways of thinking? (2) What hermeneutic meanings emerge from students' interpretations of ε, δ, and the relationships embedded within the $\varepsilon - \delta$ definition? (3) What epistemic obstacles and productive meanings characterize students' interpretations?

2. METHOD

2.1. Research Design

This study employed a qualitative hermeneutic design to investigate how students construct the mathematical meaning of the $\varepsilon - \delta$ definition from their existing ways of thinking [28]. A qualitative approach was considered appropriate because the focus of the study is not on measuring learning outcomes or testing hypotheses, but on interpreting the processes through which meaning emerges from students' engagement with formal mathematical structures. In this context, hermeneutics provides a powerful interpretive framework by positioning students' written and verbal responses as texts of meaning that can be analyzed to uncover underlying structures of understanding [29].

Drawing on philosophical hermeneutics, the understanding is viewed as an interpretive act that is historically and contextually situated, shaped by the interaction between the interpreter's horizon and the horizon embedded within the text [30]. Accordingly, students' responses are not treated merely as correct or incorrect answers, but as meaningful expressions that reflect their evolving interpretations of the $\varepsilon - \delta$ definition. This perspective allows the researcher to move beyond surface-level descriptions of students' reasoning toward a deeper exploration of how mathematical meaning is constructed [31].

Central to this design is the hermeneutic circle, an iterative process in which interpretation moves recursively between parts and wholes. Individual expressions—such as students' explanations of ε, δ, or their perceived relationship—are interpreted in relation to the overall conceptual structure of the $\varepsilon - \delta$ definition, while the meaning of the whole is simultaneously refined through engagement with these local interpretations [32]. Through this recursive movement, the analysis enables a fusion of horizons, in which students' intuitive and experiential understandings are brought into dialogue with the formal mathematical structure of the definition [33]. By adopting this hermeneutic design, the study conceptualizes students' ways of thinking not as static cognitive states, but as dynamic interpretive resources that mediate the construction of mathematical meaning. This approach aligns with qualitative traditions in mathematics education that emphasize meaning-making, interpretation, and the situated nature of understanding, thereby providing a robust methodological foundation for examining how formal concepts such as the $\varepsilon - \delta$ definition become meaningful to learners [34].

2.2. Participant Selection

The participants were 28 undergraduate students enrolled in Differential Calculus and Introductory Real Analysis courses at a university. Purposive selection was used to identify six focal participants for task-based interviews and detailed analysis of written responses, representing diverse ways of thinking patterns identified in the written responses, including symbolic-dominant, geometric, and misconception-heavy profiles. Participants are anonymized using codes such as **S1–I** (Student 1, Interview) and **S3–W** (Student 3, Written response).

.

2.3. Data Collection Instruments

Three sources of qualitative data were collected. First, students completed a set of written tasks designed to elicit their interpretations of ε, δ, distance, and dependency structures. Second, semi-structured task-based interviews were conducted to deepen the interpretation of students' mathematical meanings. Third, researcher reflective memos were maintained throughout the analytic process to document evolving interpretive insights and support reflexivity.

The written task consisted of open-ended problems requiring students to interpret and explain the $\varepsilon - \delta$ definition rather than merely reproduce its formal statement. Participants were asked to describe the meanings of ε and δ, explain the relationship between the two quantities, interpret the inequalities involved in the definition, and justify how a chosen δ satisfies a given ε condition. These tasks were designed to elicit students' ways of thinking and their emerging interpretations of the formal definition.

The task-based interviews were semi-structured and focused on clarifying participants' written responses. Interview prompts invited students to explain their reasoning, elaborate on their interpretations of ε and δ, justify perceived relationships between the quantities, and reflect on how they understood the logical order embedded in the quantified statement $\forall\varepsilon > 0 \ \exists \ \delta > 0$. Follow-up questions were used flexibly to probe meanings that emerged during the discussion. An example prompt asked students to explain why a particular δ-value would guarantee that $|f(x) - L| < \varepsilon$ for a given ε and to describe the meaning of this relationship in their own words.

2.4. Data Analysis

Data were analyzed using a four-stage hermeneutic cycle. In the first stage, naïve reading was conducted to develop an initial holistic understanding of participants' responses without imposing predetermined categories. The second stage involved selective highlighting, during which meaning-bearing expressions related to closeness, control, shrinking, equality, and dependency were identified and annotated. In the third stage, thematic interpretation was undertaken by comparing and synthesizing recurring expressions into broader interpretive themes, including distance meaning, responsive dependency, and control meaning. In the final stage, a fusion of horizons was developed by relating students' interpretive horizons to the formal mathematical meaning of the $\varepsilon - \delta$ definition [35].

 https://doi.org/10.58421/misro.v5i3.1537

Thematic development proceeded through iterative coding and interpretation. During selective highlighting, meaningful expressions related to students' understandings of ε, δ, distance, dependency, and logical sequencing were identified and annotated. Similar expressions were clustered into provisional meaning units, which were subsequently compared across participants. Through repeated movement between individual excerpts and the complete dataset, these meaning units were refined into broader interpretive themes. The final themes were established when they demonstrated coherence across multiple data sources and contributed to an integrated understanding of how students constructed meaning of the $\varepsilon - \delta$ definition.

2.5. Credibility and Trustworthiness

Credibility and trustworthiness were established through member checking with selected interview participants, peer debriefing with mathematics education colleagues, and an audit trail documenting coding decisions and thematic development. To enhance interpretive credibility, emerging interpretations and thematic constructions were periodically discussed with two colleagues experienced in mathematics education research. These discussions served as a form of peer debriefing, allowing alternative interpretations to be considered and helping to ensure that thematic conclusions remained grounded in the data rather than individual researcher assumptions. Disagreements in interpretation were discussed until a shared understanding of the thematic meanings was reached.

2.6. Ethical Considerations

Participants were informed about the purpose of the study and their voluntary involvement, and informed consent was obtained prior to data collection. All data were anonymized using codes or pseudonyms to ensure confidentiality. Participants' responses were represented faithfully, with attention to maintaining interpretive integrity. Data were stored securely and used solely for research purposes.

3. RESULTS AND DISCUSSION

3.1. Results

The first dominant theme concerns distance meaning, in which students interpreted ε and δ as measures of closeness between input and output values. Students' prior proximity thinking developed into a more structured understanding of acceptable distance, where ε represented an allowable distance around the limit value, and δ represented a corresponding distance around the input point. This meaning was evident in statements such as *"x should stay close enough so the function result remains near the target".* One participant explained: "*ε is like a boundary for how close the result should be to the target value, and δ is a boundary so that x does not go too far from the point*" (**S1-I**). Similarly, another participant wrote: "*x must be close enough so that the function value stays near the limit*" (**S2-W**). These responses indicate that students interpreted ε and δ as boundaries regulating acceptable closeness between input and output values. The emergence of distance meaning from students' proximity thinking is further illustrated in Figure 1.

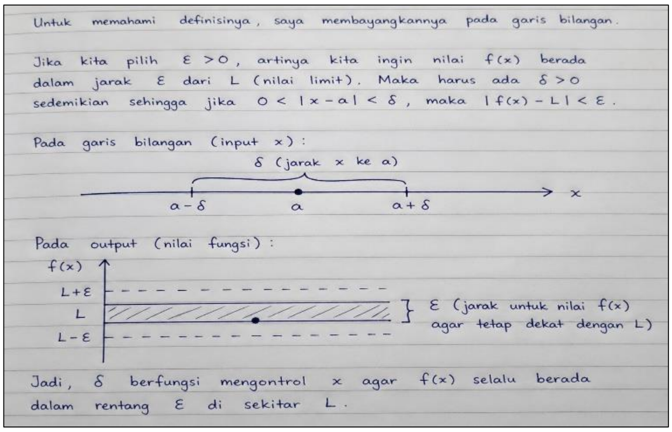

Figure 1. The written response of S2-W illustrates distance meaning in the $\varepsilon - \delta$ definition.

The written response in Figure 1 shows that the participant interpreted ε and δ as bounded measures of distance. ε was described as limiting the allowable deviation of the function value from the target value, whereas δ restricted the allowable variation of x around a given point. The response demonstrates an emerging coordination between input and output through distance-based reasoning, although the dependency between ε and δ was not yet explicitly articulated.

The second theme concerned responsive dependency. Several participants described δ as varying in response to a chosen ε rather than as an independent quantity. For example, one participant stated: "*If ε becomes smaller, then δ also needs to adjust so that the result still stays within ε*" (**S3-I**). This response suggests that the participant viewed δ as changing according to the required level of precision. However, some students continued to express equality dependency, as illustrated by the statement: "*Usually δ can be the same as ε so that both sides are balanced*" (**S4-W**). These responses indicate that students were beginning to recognize a relationship between ε and δ, although this relationship was not always interpreted in a fully relational manner. The emergence of responsive dependency is further illustrated in Figure 2.

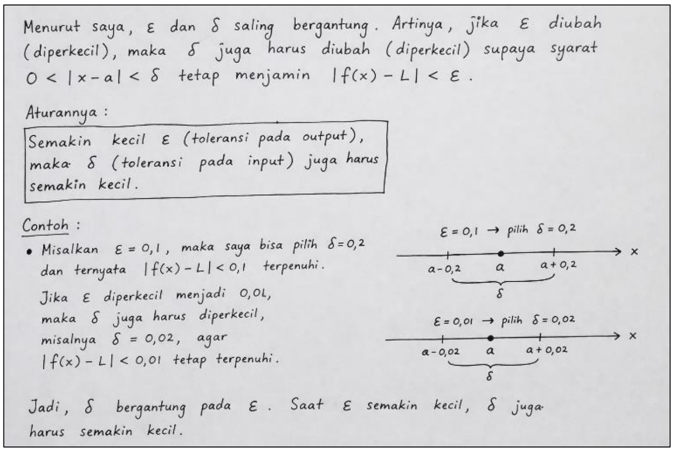

Figure 2. The written response of S4-W illustrates responsive dependency between ε and δ.

The written response in Figure 2 demonstrates that the participant interpreted ε and δ as related quantities, where changes in ε required corresponding adjustments in δ.

 https://doi.org/10.58421/misro.v5i3.1537

Numerical examples were used to illustrate how different ε-values were associated with different δ-values. Although the relationship remained informal, the response indicates an emerging awareness that δ is not fixed but varies according to the required tolerance.

The third theme was control meaning, in which students interpreted δ as a mechanism for controlling how close x must remain to ensure a desired level of output precision. One participant explained: "*δ is used to control how far x can go, so that the result does not go outside ε*" (**S5-I**). Another participant connected this idea to the logical order of the definition: "*We choose ε first because it is the target, and then we find δ so that x can meet that condition*" (**S6-I**). These responses suggest that students associated δ with the regulation of input variation and linked this regulation to the sequential structure of the $\varepsilon - \delta$ definition. The emergence of control meaning is illustrated in Figure 3.

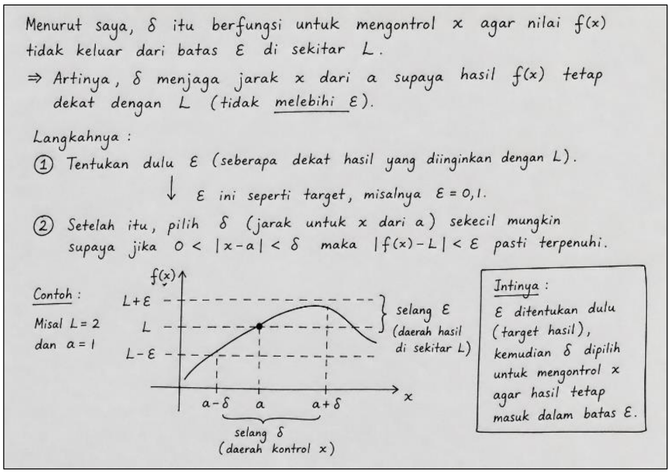

Figure 3. The written response of S5-W illustrates control meaning through the regulation of proximity.

The written response in Figure 3 indicates that the participant viewed δ as restricting the allowable variation of x so that the ε-condition remained satisfied. The response also reflects an awareness that ε functions as a target condition and that δ is subsequently selected to meet that requirement. To synthesize the thematic findings and illustrate their grounding in students' responses, Table 1 presents the relationship between the identified themes, representative excerpts, and their corresponding hermeneutic interpretations.

Table 1 summarizes the principal meanings identified across the dataset. Collectively, the findings indicate that students constructed meaning of the $\varepsilon - \delta$ definition through interconnected interpretations of distance, dependency, control, and logical sequencing. Although these meanings varied in sophistication, they reveal a progression from intuitive understandings of closeness toward increasingly structured interpretations of the formal definition.

Table 1. Thematic findings with student excerpts

Theme	Representative Student Excerpt	Hermeneutic Interpretation
Distance Meaning	"x should remain close enough so that $f(x)$ stays near the target value."	Students interpret ε and δ as measurable regions of acceptable closeness, transforming intuitive nearness into structured mathematical distance.
Responsive Dependency	"δ changes depending on the ε we want."	δ is constructed as a responsive quantity that adjusts to an externally chosen tolerance, indicating relational meaning rather than isolated symbol reading.
Equality Dependency (Obstacle)	"Usually, δ should be the same as ε so both are balanced."	Students negotiate dependency through rigid numerical symmetry, revealing an obstacle where relational flexibility is reduced to equality fixation.
Control Meaning	"δ controls how far x may move so the result stays safe."	δ is interpreted as an input-control mechanism, showing emerging understanding of regulation and strategic constraint.
Logical Sequencing	"ε must come first because we decide the target before choosing the input range."	Students begin to negotiate the temporal logic of quantifiers, connecting mathematical intention with formal sequencing.

3.2. Discussion

Reinterpreting Proximity Thinking as an Epistemic Resource

The findings suggest that proximity thinking should not be viewed solely as a misconception. Rather, students' intuitive understanding of "getting closer" served as an important starting point for constructing more structured interpretations of the $\varepsilon - \delta$ definition. The emergence of distance meaning indicates that students gradually transformed informal notions of nearness into bounded mathematical relationships involving acceptable variation around a point and a limit value.

This finding supports mathematics education research showing that intuitive reasoning can function as a productive resource for conceptual development rather than merely as an obstacle to be overcome [36]. Students' references to boundaries, closeness, and allowable distance suggest that formal understanding develops through the refinement of existing meanings. From this perspective, conceptual learning involves the reorganization of prior interpretations rather than their replacement. The findings, therefore, challenge deficit-oriented views of students' informal reasoning and highlight the importance of building upon intuitive understandings when introducing formal definitions.

The Emergence and Fragility of Responsive Dependency

A second contribution concerns students' developing understanding of the dependency between ε and δ. The findings reveal that several students interpreted δ as a quantity that responds to a previously chosen ε, indicating an emerging awareness of the relational structure underlying the formal definition. This represents an important shift from viewing mathematical symbols as isolated entities toward understanding their coordinated roles within a mathematical relationship.

At the same time, this relational understanding remained fragile. The persistence of equality dependency demonstrates that some students continued to reduce dependency on numerical symmetry, assuming that δ should be equal or proportional to ε. Similar tendencies have been reported in studies of parameter variation and functional dependency, where students often rely on balance-based reasoning when coordinating related quantities [37]. Rather than representing a simple misconception, equality dependency may be interpreted as an intermediate form of reasoning in which students recognize a relationship but have not yet differentiated dependency from numerical equivalence. The coexistence of responsive dependency and equality dependency, therefore, illustrates the negotiated and evolving nature of meaning construction.

Control Meaning and Logical Sequencing

The findings also reveal the emergence of control meaning, in which students interpreted δ as a mechanism for regulating input variation in order to satisfy a specified ε-condition. This interpretation moves beyond static symbol manipulation and reflects an emerging understanding of the functional role of δ within the $\varepsilon - \delta$ definition.

Importantly, control meaning was closely connected to students' interpretations of logical sequencing. Participants who described δ as a response to a target condition often recognized that ε must be established first before an appropriate δ can be selected. This finding aligns with research showing that students' difficulties with formal definitions frequently arise from challenges in interpreting quantifier order rather than from difficulties with notation itself [38]. Nevertheless, many participants continued to experience the quantified statement as a static symbolic expression, suggesting that the logical structure of the definition had not yet been fully internalized. These findings indicate that understanding quantifier sequencing remains a critical aspect of learning the $\varepsilon - \delta$ definition.

Hermeneutic Contribution to Meaning Construction

Beyond the specific themes identified, the findings illustrate how mathematical meaning emerges through interpretation. The progression from distance meaning to responsive dependency and control meaning suggests that students do not simply acquire the $\varepsilon - \delta$ definition as a completed formal structure. Instead, they construct meaning through repeated reinterpretations of their existing ways of thinking.

This process can be understood through the hermeneutic circle, in which understanding develops through continuous movement between individual expressions and broader conceptual structures [39]. Students' interpretations of ε, δ, dependency, and control contributed to their understanding of the definition as a whole, while their developing understanding of the whole simultaneously reshaped the meanings assigned to its individual components. The findings, therefore, support the view that learning formal mathematics involves an interpretive reconstruction of meaning rather than a linear accumulation of knowledge [40].

The results also suggest a form of what Gadamer describes as a fusion of horizons [41]. Students' intuitive experiences of closeness, adjustment, and regulation were progressively brought into dialogue with the formal mathematical structure of the $\varepsilon - \delta$ definition [42]. Rather than being discarded, these prior understandings were reorganized and integrated into more structured mathematical meanings. This perspective helps explain why conceptual development appeared gradual, non-linear, and occasionally contradictory, with productive and limiting interpretations often coexisting [43].

Theoretical and Practical Implications

The findings may also be interpreted through the lens of cognitive approaches such as APOS theory [44]. From an APOS perspective, students' movement from proximity thinking toward responsive dependency and control meaning can be viewed as a progression from actions and processes toward increasingly coordinated conceptual structures. However, whereas APOS primarily explains the cognitive mechanisms underlying concept construction, the present study focuses on the interpretive meanings through which students make sense of formal mathematical relationships. In this regard, the hermeneutic perspective complements cognitive approaches by illuminating how meaning is assigned before mathematical relationships become fully organized as formal concepts.

Theoretically, this study extends ways-of-thinking research by providing a hermeneutic account of mathematical meaning construction. The findings demonstrate that students' ways of thinking function not only as cognitive patterns but also as interpretive resources through which formal concepts become meaningful. This contribution enriches ongoing discussions concerning the relationship between intuition, interpretation, and conceptual development in mathematics education [45].

Practically, the findings suggest that instruction should explicitly address notions of distance, dependency, and control when introducing the $\varepsilon - \delta$ definition. Learning activities that require students to vary ε and construct corresponding δ-values may help strengthen their understanding of responsive dependency. Similarly, graphical and dynamic representations may support students in visualizing how δ regulates input variation to satisfy an ε-condition. In addition, the logical sequencing embedded in the quantified statement should be treated as an explicit object of classroom discussion rather than an implicit convention. Encouraging students to justify why ε precedes δ may promote deeper engagement with the relational and logical structure of the definition.

The reconstructed meanings identified in this study provide not only a hermeneutic account of meaning construction but also an empirical basis for future studies on didactic transposition and didactic organization of the $\varepsilon - \delta$ definition. The implications of these findings for ATD and TDT are discussed in the following section.

Implications for Didactic Transposition

The findings also have implications for the Anthropological Theory of the Didactic (ATD) and Theory of Didactic Transposition (TDT) [46]. From an ATD perspective, students' reconstructed meanings of distance, responsive dependency, and control can be interpreted as emerging praxeological elements that mediate their engagement with the ε–δ definition [47]. These meanings reveal how learners organize and justify mathematical activity when reasoning about limits, thereby providing insight into the relationship between institutional mathematical knowledge and students' personal understandings.

From the perspective of TDT, the findings contribute to identifying forms of knowledge that may serve as intermediate objects in the transformation of scholarly knowledge into knowledge to be taught [48]. The results suggest that concepts such as distance meaning, responsive dependency, and control meaning should not be regarded merely as informal interpretations, but as meaningful resources that can support the didactic reconstruction of the $\varepsilon - \delta$ definition. In particular, the identified themes offer a basis for specifying which conceptual relationships require explicit attention during instructional design.

Furthermore, the persistence of equality dependency highlights areas where didactic transposition may require additional epistemological support [49]. While students often recognize that ε and δ are related, they frequently interpret this relationship through numerical symmetry rather than responsive dependency [50]. This finding suggests that instructional designs should deliberately create situations in which students experience the functional and asymmetric nature of the relationship between ε and δ. Consequently, the reconstructed meanings identified in this study provide an empirical foundation for future work aimed at developing knowledge to be taught within the framework of ATD and TDT. Rather than beginning directly from the formal definition, instructional design may benefit from building upon the interpretive meanings that students already construct, thereby facilitating a more coherent transition from intuitive understanding to institutionalized mathematical knowledge.

4. CONCLUSION

This study examined how undergraduate students construct the mathematical meaning of the $\varepsilon - \delta$ definition through a hermeneutic interpretation of their existing ways of thinking. Rather than viewing students' initial reasoning patterns as misconceptions to be eliminated, the findings demonstrate that proximity thinking, symbolic interpretation, and dependency reasoning function as meaningful interpretive resources through which formal understanding gradually emerges. The analysis revealed three interconnected meanings distance meaning, responsive dependency, and control meaning, which collectively describe how students negotiate the relationship between intuitive understanding and formal mathematical structure.

The study contributes to ways-of-thinking research by extending the analysis beyond the identification of reasoning patterns toward an interpretation of how mathematical meaning is constructed. Through a hermeneutic lens, understanding is shown to develop through an iterative process of reinterpretation in which students continuously refine and reorganize existing meanings rather than replacing them through abrupt conceptual change. This perspective provides a complementary account to cognitive approaches by emphasizing the interpretive dimension of mathematical learning.

From a pedagogical perspective, the findings suggest that instruction on the $\varepsilon - \delta$ definition should explicitly engage students' intuitive notions of closeness, dependency, and control as starting points for formalization. Learning activities that encourage students to explore the relationship between ε and δ, justify quantifier sequencing, and visualize input–output constraints may support deeper conceptual understanding of formal limit definitions. Rather than presenting the $\varepsilon - \delta$ definition solely as a symbolic procedure, instruction can be designed to foreground the meanings that students construct while coordinating distance, dependency, and logical sequencing.

The findings also have implications for didactic theories, particularly the Anthropological Theory of the Didactic (ATD) and the Theory of Didactic Transposition (TDT). The reconstructed meanings identified in this study provide an epistemic basis for determining what mathematical meanings should be transformed into knowledge to be taught. Distance meaning, responsive dependency, and control meaning may therefore serve as intermediate epistemic objects that connect students' personal understandings with institutional mathematical knowledge. In this sense, the study contributes to ongoing discussions concerning the transformation of learners' meanings into teachable mathematical organizations.

Several limitations should be acknowledged. The study was conducted with a relatively small group of undergraduate students within a specific instructional context, and the findings should therefore be understood as context-dependent interpretations rather than universally generalizable stages of learning. In addition, the hermeneutic approach emphasizes interpretive depth, meaning that alternative interpretations of the same data remain possible despite the use of member checking and peer debriefing procedures.

Future research may examine how the meanings identified in this study develop across different educational levels, institutional contexts, or instructional interventions. Further studies may also integrate hermeneutic perspectives with cognitive frameworks such as APOS theory, Concept Image–Concept Definition theory, or other models of conceptual development to provide a more comprehensive account of learning. More broadly, this study highlights the value of hermeneutic approaches in mathematics education by demonstrating that students' informal understandings are not merely obstacles to be corrected, but important resources for constructing meaningful engagement with formal mathematical ideas. Such an orientation may contribute to the design of more meaningful and accessible mathematics learning experiences for students and educators.

ACKNOWLEDGEMENTS

The authors would like to thank the participating students for their valuable contributions and willingness to share their perspectives throughout this study. Appreciation is also extended to colleagues in mathematics education at Universitas Singaperbangsa Karawang for their constructive feedback and academic support.

REFERENCES

[1] A, Howard., & Herr, *Calculus with analytic geometry*. New York: Wiley, 1998.

[2] Bartle, R. G., & Sherbert, D. R, *Introduction to real analysis* (Vol. 2). New York: Wiley, 2000.

[3] Boero, P, Analyzing the transition to epsilon-delta Calculus: A case study. In *CERME 9-Ninth Congress of the European Society for Research in Mathematics Education*, pp. 93-99, 2015.

[4] Adiredja, A. P, The pancake story and the epsilon–delta definition. *Primus*, *31*(6), 662-677, 2021.

[5] Caglayan, G, Math majors' visual proofs in a dynamic environment: The case of limit of a function and the ϵ–δ approach. *International journal of mathematical education in science and technology*, *46*(6), 797-823, 2015.

[6] Cornu, B, Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153–166). Dordrecht: Kluwer Academic Publishers, 1991.

[7] Kim, D. J., & Lim, W. The relative interdependency of colloquial and mathematical discourses regarding the notion and calculations of limit: An evidence-based cross-cultural study. *International Journal of Science and Mathematics Education*, *16*(8), 1561-1579, 2018.

[8] Kim, D., Kang, H., & Lee, H. Two different epistemologies about limit concepts. International Education Studies, 8(3), 138–145, 2015.

[9] Rasmussen, C., Marrongelle, K., & Borba, M, Research on calculus: What do we know and where do we need to go? *ZDM Mathematics Education*, 46(4), 507–515, 2014.

[10] Rasmussen, C., Marrongelle, K., & Borba, M, Research trends in undergraduate mathematics education. *Journal for Research in Mathematics Education*, 5, 2021.

[11] Weber, K, Students' difficulties with proof and quantifiers. *Journal of Mathematical Behavior*, 43, 50–64, 2016.

[12] Williams, S. R, Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, 22(3), 219–236, 1991.

[13] Roh, K. H. An empirical study of students' understanding of a logical structure in the definition of limit via the ε-strip activity. Educational Studies in Mathematics, 73(3), 263-279, 2010.

[14] Prihandhika, A., & Ramadani, I, Interpreting the ε–δ Definition: Hermeneutic Insights into Students' Ways of Thinking. *Riemann: Research of Mathematics and Mathematics Education*, *8*(1), 341-358, 2026.

[15] Prihandhika, A., & Sopiany, H. N, Procedural Dominance in Students' Reasoning on the Limit Definition: Insights from a Ways of Thinking Framework. *Journal of Mathematics Instruction, Social Research and Opinion*, *4*(4), 1313-1324, 2025.

[16] Kidron, I., & Tall, D. The role of embodiment and symbolism in the potential and actual infinity of the limit process. *Educational Studies in Mathematics*, 2015.

[17] Gucler, B. The role of symbols in mathematical communication: The case of the limit notation. *Research in Mathematics Education, 16*(3), 251–268, 2014, https:// doi. org/ 10. 1080/ 14794 802

[18] Gucler, B. Examining the discourse on the limit concept in a beginning-level calculus classroom. *Educational Studies in Mathematics, 82*(3), 439–453. https:// doi. org/ 10. 1007/ s10649- 012- 9438-2, 2013.

[19] Fernández, E, The students' take on the ε–δ definition of a limit. *Journal of Mathematical Behavior*, 23(2), 215–236, 2004.

[20] Fernández-Plaza, J. A., & Simpson, A. Three concepts or one? Students' understanding of basic limit concepts. Educational Studies in Mathematics, 93(3), 315-332, 2016.

[21] Sierpinska, A, *Understanding in mathematics*. London: Falmer Press, 1994.

[22] Roh, K. H, An empirical study of students' understanding of a logical structure in the definition of limit. *Journal of Mathematical Behavior*, 29(2), 97–115, 2010.

[23] Hitt, F., & Dufour, S. Introduction to calculus through an open-ended task in the context of speed: representations and actions by students in action. *ZDM–Mathematics education*, *53*(3), 635-647, 2021.

[24] Tall, D, How humans learn to think mathematically: Exploring the three worlds of mathematics. Cambridge: Cambridge University Press, 2013.

[25] Tall, D., & Vinner, S, Concept image and concept definition in mathematics. *Educational Studies in*

Mathematics, 12(2), 151–169, 1981.

[26] Prihandhika, A., & Perbowo, K. S, The review of concept image and concept definition: A hermeneutic phenomenological study on the derivative concepts. *International Journal of Didactic Mathematics in Distance Education*, *1*(1), 13-23, 2024.

[27] Brown, T, *Mathematics education and language: Interpreting hermeneutics and post-structuralism*. Dordrecht: Springer Netherlands, 2002.

[28] Carl, M, Understanding mathematical texts: a hermeneutical approach. *Synthese*, *200*(6), 524, 2022.

[29] Creswell, J. W, *Qualitative inquiry and research design: Choosing among five approaches* (3rd ed.). Thousand Oaks, CA: Sage, 2013.

[30] Ellis, A. B., Ozgur, Z., & Kulow, T, Quantitative reasoning in mathematics learning: Leveling the playing field. *Journal for Research in Mathematics Education*, 47(2), 131–164, 2016.

[31] Jahnke, H. N, History in mathematics education. A hermeneutic approach. In *Mathematics & mathematics education: Searching for common ground* (75-88). Dordrecht: Springer Netherlands, 2014.

[32] Markle, J, Enactive hermeneutics as an interpretive framework in the mathematics classroom. *Philosophy of Mathematics Education Journal*, *37*, 1-27, 2021.

[33] Ricoeur, P, History and hermeneutics. In *Philosophy of History and Action: Papers Presented at the First Jerusalem Philosophical Encounter December 1974* (3-20). Dordrecht: Springer Netherlands, 1978.

[34] Skinner, Q, Hermeneutics and the Role of History. *New Literary History*, *7*(1), 209-232. 2(1), 1–12, 1975.

[35] Gadamer, H.-G, *Truth and method* (2nd rev. ed., J. Weinsheimer & D. G. Marshall, Trans.). London: Continuum, 2004.

[36] Mahadewsing, R., Getrouw, D., & Calor, S. M. Prior knowledge of a calculus course: The impact of prior knowledge on students' errors. *International Electronic Journal of Mathematics Education*, *19*(3), em0786, 2024.

[37] Carpenter, T. P. Conceptual knowledge as a foundation for procedural knowledge. In *Conceptual and procedural knowledge*, Routledge, (pp. 113-132), 2013.

[38] Ricoeur, P, *Interpretation theory: Discourse and the surplus of meaning*. Fort Worth, TX: Texas Christian University Press, 1976.

[39] Lesh, R., & Harel, G. Problem solving, modeling, and local conceptual development. *Mathematical thinking and learning*, *5*(2-3), 157-189, 2003.

[40] Braithwaite, D. W., & Sprague, L. Conceptual knowledge, procedural knowledge, and metacognition in routine and nonroutine problem solving. *Cognitive Science*, *45*(10), e13048, 2021.

[41] Debesay, J., Nåden, D., & Slettebø, Å. How do we close the hermeneutic circle? A Gadamerian approach to justification in interpretation in qualitative studies. *Nursing inquiry*, *15*(1), 57-66, 2008.

[42] Hintikka, J. Gadamer: squaring the hermeneutical circle. *Revue internationale de philosophie*, *54*(213 (3), 487-497, 2000.

[43] Sierpinska, A. Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics*, *18*, 371–397, 1987.

[44] Dubinsky, E., & McDonald, M. A. APOS: A constructivist theory of learning in undergraduate mathematics education research. In *The teaching and learning of mathematics at university level: An ICMI study*, Dordrecht: Springer Netherlands, 275-282, 2001.

[45] Harel, G., Selden, A., & Selden, J. Advanced mathematical thinking: Some PME perspectives. In *Handbook of research on the psychology of mathematics education*, Brill, 147-172, 2006.

[46] Chevallard, Y., & Bosch, M. Didactic transposition in mathematics education. In *Encyclopedia of mathematics education*, Cham: Springer International Publishing, 214-218, 2020.

[47] González-Martín, A. S. Using Tools from ATD to Analyse the Use of Mathematics in Engineering Tasks: Some Cases Involving Integrals. In *Advances in the Anthropological Theory of the Didactic*, Cham: Springer International Publishing, 307-315, 2022.

[48] Topphol, V. The didactic transposition of the fundamental theorem of calculus. *Journal of Research in Mathematics Education*, *12*(2), 144-172, 2023.

[49] Biehler, R. Reconstruction of meaning as a didactical task: The concept of function as an example. In

 https://doi.org/10.58421/misro.v5i3.1537

meaning in mathematics education, New York, NY: Springer US, 61-81, 2005.

[50] Gruschka, A. How we can and why we have to reconstruct teaching. *International Journal for Lesson and Learning Studies*, *7*(2), 85-97, 2018.